External Memory Minimum Spanning Trees

Dominik Schultes

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angefertigt unter Betreuung von Priv. Doz. Dr. Peter Sanders, Max-Planck-Institut für Informatik

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Saarbrücken, im August 2003

Abstract

While in the last years much theoretical work about external memory minimum spanning trees was done, the practical realization of the designed algorithms was neglected. It is the goal of my Bachelor thesis to fill this gap, i.e., we will show that the computation of minimum spanning trees of very large graphs is possible efficiently not only in theory but also in practice.

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Chapter 1

Introduction

1.1 Minimum Spanning Trees

Finding a minimum spanning tree is the graph theoretical notation of a quite natural problem: we want to connect several objects — not necessarily directly — with each other and the total costs should be as low as possible. As example we could take a computer network: several computers have to be linked so that every pair of computers can communicate with each other regardless of possible intermediate stations. The expense of the used wires should be minimal.

In graph theory the problem can be formalized easily. In a connected, undirected and weighted graph $G = (V, E)$ with n vertices and m edges a spanning tree $T = (V, F)$ is a connected and acyclic subgraph of G that contains all nodes of G. The weight of a spanning tree is the sum of the weights of all edges: $w(T) = \sum_{e \in F} w(e)$. A minimum spanning tree (MST) is a spanning tree with minimum weight.

The problem of finding a MST is interesting because of several reasons. First, it appears in praxis. Second, the problem itself is interesting from a theoretical point of view, and third, the solution of it can be used in order to solve other problems in graph theory, for instance, to find an approximate solution for the travelling salesperson problem.

There are two well known algorithms that can be used to detect a MST in $O(n^2)$ (if the input is given as adjacency matrix) resp. in $O(m \log m)$ time (if the input is given as adjacency lists), Prim's algorithm and Kruskal's algorithm.

1.2 External Memory Model

In the main, our considerations refer to a simple model that focuses on two levels of the memory hierarchy, the internal memory (main memory) and the external memory (hard disks). Due to physical reasons (the access head has to find the right position) a disk access takes comparatively much time, the transfer itself plays a tangential role [MSS03, p. 3]. Hence, it is reasonable to deal with blocks of data instead of single items. The *internal memory size* is denoted as M and the *block size* as B [AV88].

The main reason why we have to use external memory algorithms is the simple fact that we do not possess enough internal memory since main memory is C times more expensive than hard disks; we assume that $C \approx 200$ and that consequently a balanced system consists of internal and external memory in the ratio of 1:200.

1.3 External Memory Minimum Spanning Trees

The External Memory MST problem deals with very large graphs, i.e., the graphs are so large that not all information which are required for the computations fit in internal memory. Finding a minimum spanning tree of such a graph is more difficult than the internal case since you have to ensure that you take advantage of both spacial and temporal locality as much as possible. Neither Prim's nor Kurskal's algorithm has been designed with this requirement in mind. Hence, in the worst case the processing of each node resp. edge requires one external memory access. Therefore, we need a different approach.

Chapter 2

Dense Graphs

If we deal with large dense graphs, we are confronted with the following situation: there are so many edges that they do not fit in internal memory so that we cannot apply an internal algorithm — but there are so few nodes that we can afford some internal memory for each node, in other words, the internal memory is in $O(n)$. Generally, this is the precondition for a semi-external algorithm. In our case, we can use a semi-external version of Kruskal's algorithm [Ski98] in order to find a minimum spanning tree of a dense graph. We sort the list of edges (by weight) in external memory and apply Kruskal's algorithm so that the sorted edges have to be scanned once. Hence, we need $O(\text{sort}(m)) + O(\text{scan}(m)) = O(\text{sort}(m))$ I/Os.

A union-find data structure is essential for Kruskal's algorithm. Basically, we need for each node a reference to the parent node. Additionally, we want for each canonical node of a set (= root of a tree) a rank that is used to perform the union operations in such a way that the trees do not degenerate. Thus, the size of the union-find data structure depends only linearly on the number of nodes so that it can be kept in internal memory on the above mentioned precondition.

In order to keep the memory usage per node small, we store only the height of a tree (instead of the size) as rank. Consequently, we use the union-by-height strategy [OW96], i.e., when we unite two trees, we make the shorter one a subtree of the taller one; if both trees have the same height, an arbitrary one becomes the subtree of the other one, whose height is increased. This strategy guarantees that the height of each tree does not exceed $\log n$ so that $\log \log n$ bits are sufficient to store the rank. After each find operation path compression is applied. (It is possible that the height of one tree is reduced due to path compression. In this case the stored height is *not* adjusted. The expense of a correction would exceed the advantage of an exact value by far.)

Usually a dense graph is expected to consist of $O(n^2)$ edges. As we want to use the distinction between "dense" and "sparse" in order to specify the applicability of the semi-external algorithm, we consider graphs with less edges as dense, too. With this in mind, the boundary between "dense" and "sparse" can be computed with the help of the estimation of Section 1.2 that a balanced system consists of internal and external memory in the ratio of $1{:}C$. Principally, we can process only graphs that fit in external memory. If the edges are given as a list, we have to store for each edge the source vertex, the target vertex and the weight. (To store the edges in adjacency lists or in an adjacency matrix would require more memory.) In order to keep this calculation simple, we assume that a node identifier, a weight and a rank in the union-find data structure needs one memory unit each. Hence, we need $3m$ memory units to store the graph, while the size of the external memory is $C \cdot M$. This leads to $\frac{3}{C}m \leq M$. For each node two memory units are allocated in the union-find data structure, a reference to the parent node and the rank. As we want to treat the case that the union-find data structure does not fit in internal memory, we obtain the constraint that $2n > M$. If we combine these inequalities, we get $\frac{m}{n} < \frac{2}{3}C$. Hence, we consider a graph with at least $\frac{2}{3}Cn$ edges as dense and a graph with less edges as sparse. Furthermore, we can state that dense graphs (within this scope) can be processed by our semi-external algorithm.

Actually, a more sophisticated calculation could draw the line even at a smaller average vertex degree.

Chapter 3

Sparse Graphs

3.1 General Approach

If we deal with large sparse graphs, we can keep neither the nodes nor the edges in internal memory. Hence, the semi-external version of Kruskal's algorithm introduced in chapter 2 cannot be applied directly so that we need an external algorithm. Our basic goal is *to reduce the number of nodes* of the original graph G until the union-find data structure of the remaining nodes (graph G') fits in internal memory so that we can apply the algorithm of chapter 2 to G'. During the node reduction phase we obtain a part of a minimum spanning tree of G, which is combined with a minimum spanning tree of G' in order to get a complete MST of G. The following lemma [JaJ92, p. 223] provides the foundation in order to achieve this goal.

Lemma 1. *Let $V = \bigcup V_i$ be an arbitrary partition of V with the corresponding subgraphs $G_i = (V_i, E_i)$, where $1 \leq i \leq t$. For each i, there exists one minimum-weight edge e_i connecting a vertex in V_i to a vertex in $V \setminus V_i$ that belongs to a minimum spanning tree of the graph $G = (V, E)$.*

Proof. For an arbitrary subset $V_i \subset V$, let the set $R_i := \{(r, s) \in E \mid r \in V_i \wedge s \in V \setminus V_i\}$ contain all edges that lead from one vertex in V_i to a vertex out of V_i, and let the set $S_i \subseteq R_i$ contain the shortest edges of R_i, i.e., $S_i := \{e \in R_i \mid w(e) = \min_{e' \in R_i} w(e')\}$. We assume that there is a minimum spanning tree $T = (V, F)$ with a subset $V_i \subset V$ so that $F \cap S_i = \emptyset$, i.e., T does *not* contain any of the shortest edges that leave V_i.

Let $(u', v') \in S_i$ be an arbitrary shortest edge that connects V_i with $V \setminus V_i$. We add (u', v') to F and obtain a cycle that contains (u', v') and a different edge $(u, v) \in R_i \setminus S_i$, i.e., an edge that leaves V_i, but does *not* belong to the shortest ones. We remove (u, v) and get another spanning tree T' with $w(T') < w(T)$ since $w((u', v')) < w((u, v))$. This contradicts the assumption that T is a minimum spanning tree. □

3.2 Boruvka's Algorithm

The most known node reduction algorithm that uses Lemma 1 is *Boruvka*'s. At the beginning the partition of $G = (V, E)$ consists of single nodes. For each node a minimum-weight edge incident to it is found so that the selected edges do not form a cycle. These edges are added to the tree $T = (V, F)$, which initially contains no edges and eventually will be a MST of G. The next iteration deals with the partition $V = \bigcup V_i$, where each V_i is a connected component of the subgraph $G' = (V, F)$. Hence, one iteration, called a Boruvka step, consists of the following substeps [Liu01]:

- for each node, find and mark an appropriate minimum-weight edge incident to it;
- determine the connected components formed by the marked edges;
- replace each connected component by a single (super-)vertex, in other words, relabel the edges in such a way that the vertex IDs are replaced with the IDs of the appropriate connected components;
- optionally, eliminate the self-loops and multiple edges created by these contractions.

The algorithm terminates when the remaining graph can be processed by the semi-external algorithm or — if Boruvka's algorithm is used to find a complete MST — when $|F| = n - 1$.

In a basic version of this algorithm, we need $O(\text{sort}(m))$ I/Os for one Boruvka step in order to reduce the number of vertices by a constant factor. $O(\log(n/M))$ steps are required so that the remaining nodes fit in internal memory M. This results in $O(\text{sort}(m) \cdot \log(n/M))$ [CGG+95]. A top-down variant with the same I/O complexity is presented in [ABW02].

An improved version [MSS03, p. 80–81] uses $O(\text{sort}(m) \cdot \max\{1, \log\log(nB/m)\})$ I/Os in order to find a MST. This improvement is achieved by combining several steps into supersteps, where each superstep still needs $O(\text{sort}(m))$ I/Os, but reduces more nodes than the basic step. A randomized algorithm, presented in [ABW02], uses $O(\text{sort}(m))$ I/Os in the expected case.

In spite of the good asymptotic behaviour, an implementation of Boruvka's algorithm probably would lead to high constants in the running time. Hence, we will use a different algorithm that has the same general approach.

3.3 Sibeyn and Meyer's Algorithm

3.3.1 Informal Description

In a way Sibeyn and Meyer's algorithm [SM] is a variant of Boruvka's algorithm. During one step only the minimum-weight edge incident to a node in the last subset V_t is determined and added to the tree instead of finding a shortest edge for each subset V_i.

The input is given as a set of m edges (u, v, c), where u is the source vertex, v the target and c the weight. As the graph is undirected an edge (u, v, c) implies that there is also an edge (v, u, c), although it is not listed explicitly. The edges are stored in adjacency lists; each edge is stored only once, namely in the list of the node with the higher identifier; self loops are thrown away since they are irrelevant. The chosen data structure has the feature that only the list of the last node definitely contains all edges incident to it; but that is quite enough as we concentrate on the last node during each step.

In order to get a new partition after each step, the graph is shrunk. The last node is merged with the target vertex of the shortest edge incident to the last node, i.e., the target vertex adopts all edges from the last node and the last node stops existing. Self loops that are created by this action are thrown away. This merging corresponds with the union of the last node and the target vertex to a new subset of V that is part of the partition of the graph. Due to merging edges are relabeled because the former source vertex is replaced with the target vertex of the shortest edge. In order to be able to restore the original endpoints when an edge is added to the MST, the original labels are saved at the beginning so that each adjacency list contains edges (v, c, e_1, e_2), where v is the target, c the weight and e_1, e_2 the original endpoints.

3.3.2 Pseudo Code

Input: a set E of edges (u, v, c) that defines a connected, undirected and weighted graph G with n nodes,
n', the number of nodes that should remain
Output: a set $T \subseteq E$ that defines (a part of) a minimum spanning tree of G

let π **be** a random permutation over $\{1, \ldots, n\}$
foreach $(u, v, c) \in E$ **do**
    if $v < u$ **then** add $(\pi(v), c, u, v)$ to the list of $\pi(u)$
    else if $v > u$ **then** add $(\pi(u), c, u, v)$ to the list of $\pi(v)$;

for $u = n$ **down to** $n' + 1$ **do**
    traverse all (v, c, e_1, e_2) in the list of u and
    determine the v for which c is minimum;
    add (e_1, e_2, c) to T;
    foreach (w, c, e_1, e_2) in the list of u **do**
        if $w < v$ **then** add (w, c, e_1, e_2) to the list of v
        else if $w > v$ **then** add (v, c, e_1, e_2) to the list of w;

3.3.3 Correctness

The correctness follows directly from Lemma 1.

3.3.4 Complexity

We assume that the input does not contain any self-loop (self-loops would be eliminated anyway during the first step).

If initially the node indices are randomized (so that we get a uniform distribution), the probability that an arbitrary edge is incident to the last vertex is $\frac{1}{n} + \frac{1}{n} = \frac{2}{n}$ as it is sufficient if one of the endpoints is the last node (the case that both endpoints are the last node cannot occur due to our assumption that there is no self-loop). Hence, the expected number of edges in the list of the last node is $\frac{2}{n}m$.

Since the target vertex of the shortest edge is uniformly distributed, too, we obtain a uniform distribution over the set $\{1, \ldots, n-1\}$ after one reduction step if the edges have been uniformly distributed over $\{1, \ldots, n\}$: the probability that an endpoint of an edge is a certain node $x \in \{1, \ldots, n-1\}$ amounts to $\frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n-1} = \frac{1}{n-1}$, namely the probability according to the assumed uniform distribution over $\{1, \ldots, n\}$ plus the probability that the endpoint had been the last node and was then relabeled to x due to a reduction step.

Using these facts, we can show by induction that the expected number of edges in the list of the currently last node $u \in \{n'+1, \ldots, n\}$ is always less than (or equal to) $\frac{2}{u}m$. Therefore, we obtain [SM]

Theorem 1. *For reducing the number of nodes from n to n', the above algorithm processes an expected number of less than $\sum_{u=n'+1}^{n} \left(\frac{2}{u}m\right) \simeq 2 \cdot m \cdot (\ln n - \ln n')$ edges.*

3.3.5 Comparison with Boruvka's Algorithm

The advantages of Sibeyn and Meyer's algorithm over Boruvka's algorithm are the following:

- During one Boruvka step $2 \cdot m$ edges are processed in order to reduce the number of nodes by only a factor 2 in the worst case. For a reduction by this factor, the expected number of processed edges of Sibeyn and Meyer's algorithm is less than $2 \cdot m \cdot \ln 2 \simeq 1.39 \cdot m$.
- Relabeling the edges due to shrinking of the graph is very easy in Sibeyn and Meyer's algorithm because the new identifier need not be looked up — in contrast with Boruvka's algorithm, where the new source *and* target vertices of all edges must be looked up. Furthermore, Sibeyn and Meyer's algorithm dispenses with finding connected components.
- In Sibeyn and Meyer's algorithm the reduced graph can directly be taken as input for the semi-external version of Kruskal's algorithm since the first n' nodes are preserved. In Boruvka's algorithm it is more difficult to guarantee that the node identifiers are a sequence without gaps in order to be able to index the union-find data structure.
- In Boruvka's algorithm with full adjacency lists $2 \cdot m$ edges have to be stored in total, in Sibeyn and Meyer's algorithm only m edges are stored at any time.

3.3.6 External Realization with Buckets

In order to implement Sibeyn and Meyer's algorithm, we need one adjacency list for each node. We cannot keep all edges in internal memory at the same time, so we have to consider a reasonable disposition of the data in external memory to obtain an efficient implementation. We read only from the list of the last node, but the relabeled edges are written to arbitrary nodes (but the last). Of course we can easily read the edges of the last node blockwise, but it is difficult to write edges blockwise because we cannot afford a write buffer[1] for each node. To solve this problem, we distribute the edges to several buckets, so that we can afford a write buffer for each bucket: we have b buckets and upper bounds $u_0 < u_1 < u_2 < \ldots < u_b, u_0 = 0, u_b \geq n$,

[1] In order to save I/O operations, we must not write the apartly incoming edges immediately. Rather we gather the edges in write buffers until the disk access is worthwhile.

so that bucket $i \in \{1, \ldots, b\}$ contains the edges of the nodes with the identifiers from $u_{i-1} + 1$ to u_i in an arbitrary sequence. (According to the considerations of Section 3.3.1, "the edges of one node" means only the edges that lead to nodes with lower identifiers.)

The buckets can be used directly to write relabeled edges because we can add an edge to the appropriate bucket without worrying about assigning it to the exact node. Nevertheless, we have to worry about the exact node when we want to read the edges of the currently last node. Therefore, we read the complete last external bucket i at a single blow and distribute the edges to internal buckets so that one internal bucket contains all edges of one node. Then the edges of the nodes u_i down to $u_{i-1} + 1$ can be processed before the next external bucket $i - 1$ is loaded.

The first external bucket contains the edges of the nodes that fit in internal memory, i.e., $u_1 = n'$. So, when the second external bucket has been processed, the node reduction is completed and the first bucket contains the reduced graph and can be used as input for Kruskal's algorithm.

Figure 3.1 represents the two layers of data and the processing of the edges during the node reduction phase.

3.3.7 External Realization with a Priority Queue

Alternatively, one external priority queue [San00] can be used instead of several external and internal buckets; in this case only one external bucket is needed in order to store the edges of the first n' nodes, which will be processed by Kruskal's algorithm. The shortest edge incident to the last node is on top of the queue, followed at first by the other edges of the last node and then by the shortest edge incident to the second last node and so on. The elements in the queue are quintuples (u, v, c, e_1, e_2), where u is the source, v the target, c the weight and e_1, e_2 the original endpoints. Hence, the algorithm can be restated in the following way:

let π **be** a random permutation over $\{1, \ldots, n\}$
foreach $(u, v, c) \in E$ **do** push($(\pi(u), \pi(v), c, u, v)$);

$s := -1$;
while not pqueue.empty() **do**
    $(u, v, c, e_1, e_2) :=$ pqueue.pop();
    if $u \neq s$ **then**
        $(s, t) := (u, v)$;
        add (e_1, e_2, c) to T;
    else
        push((t, v, c, e_1, e_2));

procedure push((u, v, c, e_1, e_2))
        if $u \neq v$ **then**
            if $\max(u, v) \leq n'$ **then** bucket.push((u, v, c, e_1, e_2))
            else pqueue.push($(\max(u, v), \min(u, v), c, e_1, e_2)$);

The main advantage of using an external priority queue is the scalability: we expect good results for any kind of graphs, even for degenerated ones, for instance, graphs with a small average vertex degree containing some nodes with a very high degree. However, the realization with several buckets (3.3.6) will be faster in most cases, but can get into trouble if it has to deal with such degenerated graphs.

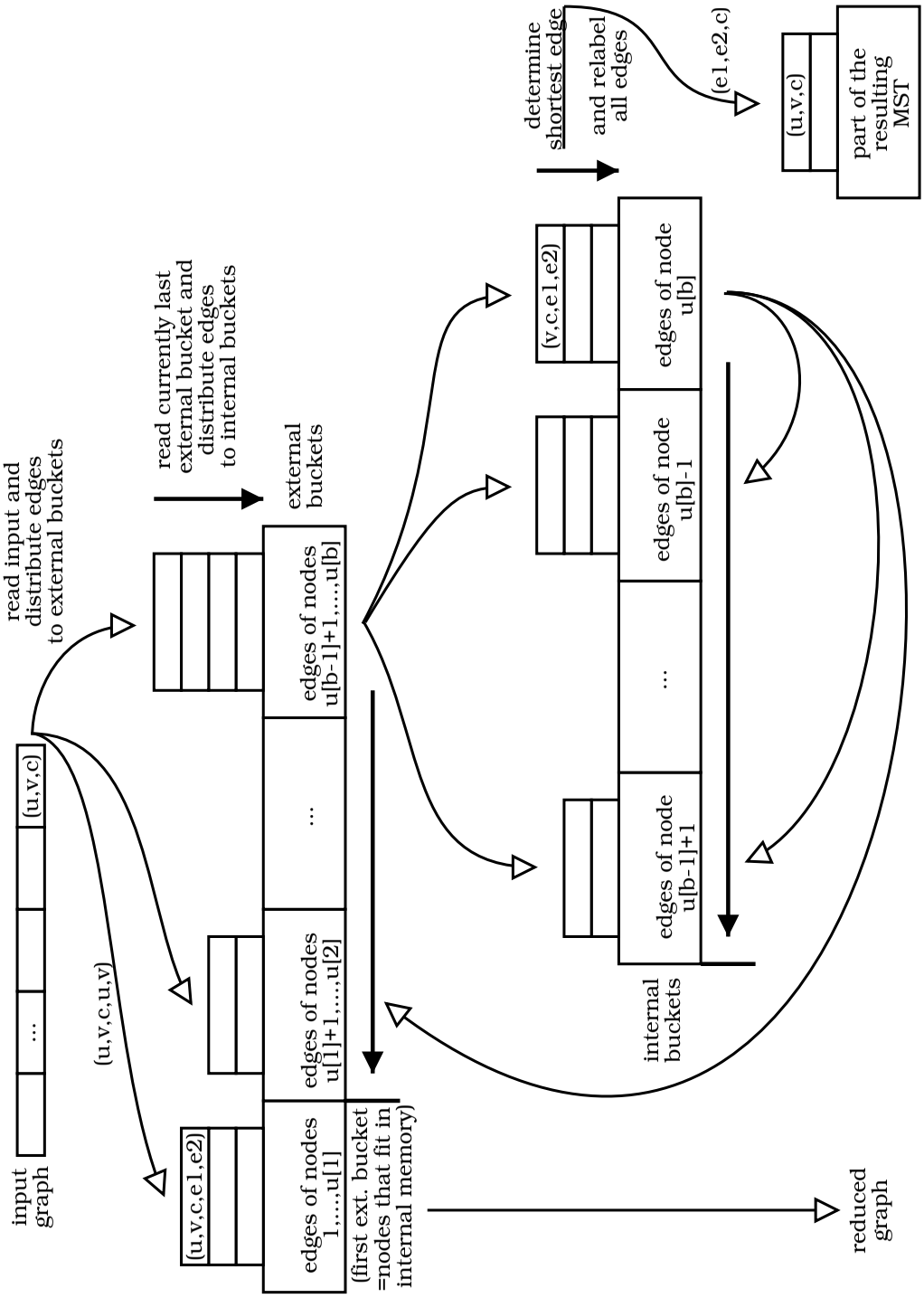


Figure 3.1: Sibeyn and Meyer's algorithm — external realization

Chapter 4

Implementation

4.1 <stxxl> Library

The implementation uses the <stxxl> library, which is developed at the Max-Planck-Institute for Computer Science. "The core of <stxxl> is an implementation of the C++ standard template library STL for external memory (out-of-core) computations, i.e., <stxxl> implements containers and algorithms that can process huge volumes of data that only fit on disks. While the compatibility to the STL supports ease of use and compatibility with existing applications, another design priority is high performance [...]: transparent support of multiple disks, variable block lengths, overlapping of I/O and computation, prevention of OS file buffering overhead." [Dem03]

4.2 Data Structures

The basic data structure is the class `Edge` that represents an edge consisting of two endpoints (called `source` and `target`) and the `weight`. Furthermore, we need a class `RelabeledEdge` that is a subclass of `Edge` and additionally contains the `original source` and the `original target`. Sometimes it is not necessary to store the source vertex because if we look at the adjacency list of one particular vertex, the source vertex is known implicitly. In this case we use a class `RelabeledEdgeWithoutSource` in order to reduce memory usage. A `RelabeledEdgeWithoutSource` consists of `target`, `weight`, `original source` and `original target`. (In order to avoid multiple inheritance `RelabeledEdgeWithoutSource` is *not* a superclass of `RelabeledEdge`.) A superclass `EdgeWithoutSource` encapsulates the common components of `Edge` and `RelabeledEdgeWithoutSource`.

The class `EdgeVector` extends the `stxxl::vector`-class and can be used to save a sequence of edges. As `EdgeVector` is a template class, it can be used for both edges and relabeled edges. The main feature of `stxxl::vector` is the storage of the data in external memory while some blocks of data stay in internal memory so that reading and writing is always done blockwise. As the subclass `EdgeVector` should be able to represent a graph, it additionally stores the number of nodes of the graph. Furthermore, there is a method `sortByWeight()` that uses `stxxl::ksort` [DS03] in order to sort the edges by weight.

Finally, the class `MST` represents a (part of a) minimum spanning tree and mainly consists of an `EdgeVector<Edge>`. There are several methods to `add` an `Edge`, a `RelabeledEdge` or a `RelabeledEdgeWithoutSource` to the MST. Polymorphism is avoided due to efficiency reasons.

Figure 4.1 summarizes these data structures.

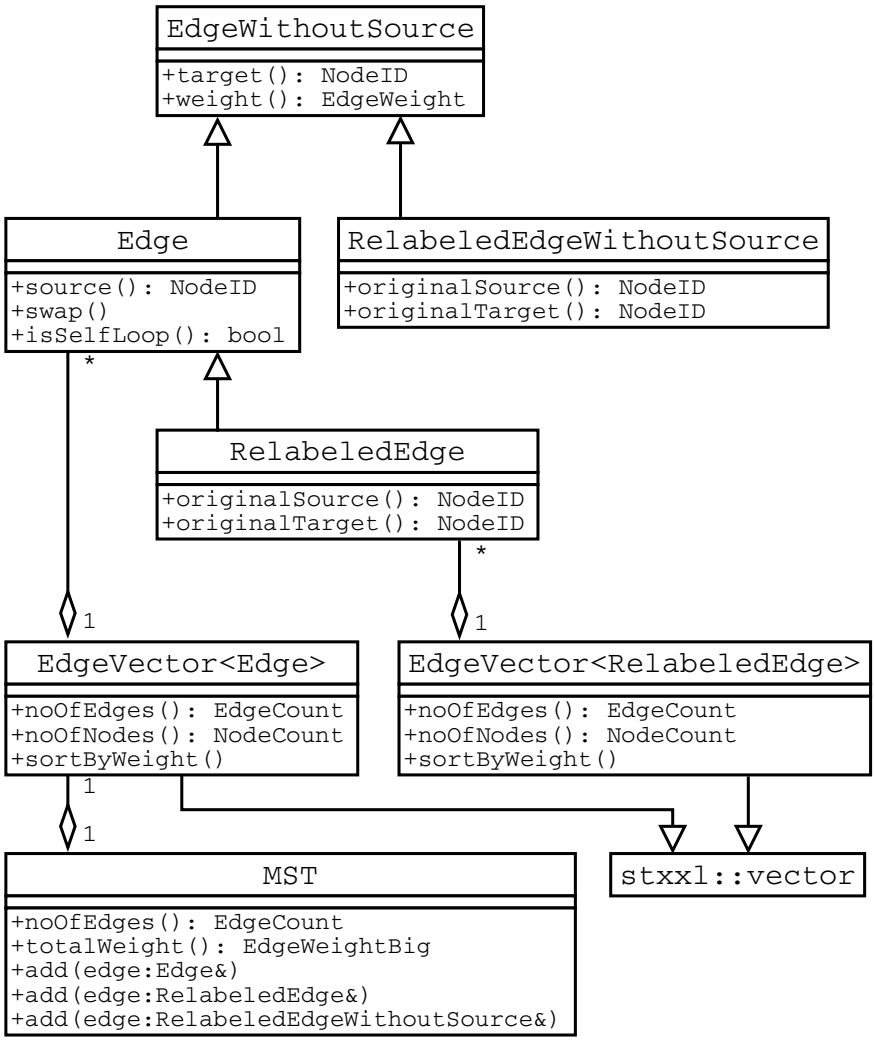


Figure 4.1: UML class diagram [BRJ99] — data structures

4.3 Base Case

We can apply the base case when the union-find data structure of all (remaining) nodes fit in internal memory. For this case the class `Kruskal` provides data structures and methods to apply a semi-external version of Kruskal's algorithm. The constructor is given a reference to a graph represented by an `EdgeVector` and a reference to the `MST`-object that stores the resulting MST. Since `Kruskal` is a template class, it can deal with both an `EdgeVector<Edge>` and an `EdgeVector<RelabeledEdge>`.

First the edges are sorted (using the `sortByWeight()`-method of `EdgeVector`) and then the edges are scanned and appropriate union-find operations are performed.

Figure 4.2 is an overview of the interface of the `Kruskal`-class.

4.4 Node Reduction with Buckets

4.4.1 External Buckets

We have to be able to add edges to an external bucket and read all edges of the currently last bucket, so the functionality of a stack is sufficient. Therefore, we use a `stxxl::stack` for each external bucket. The

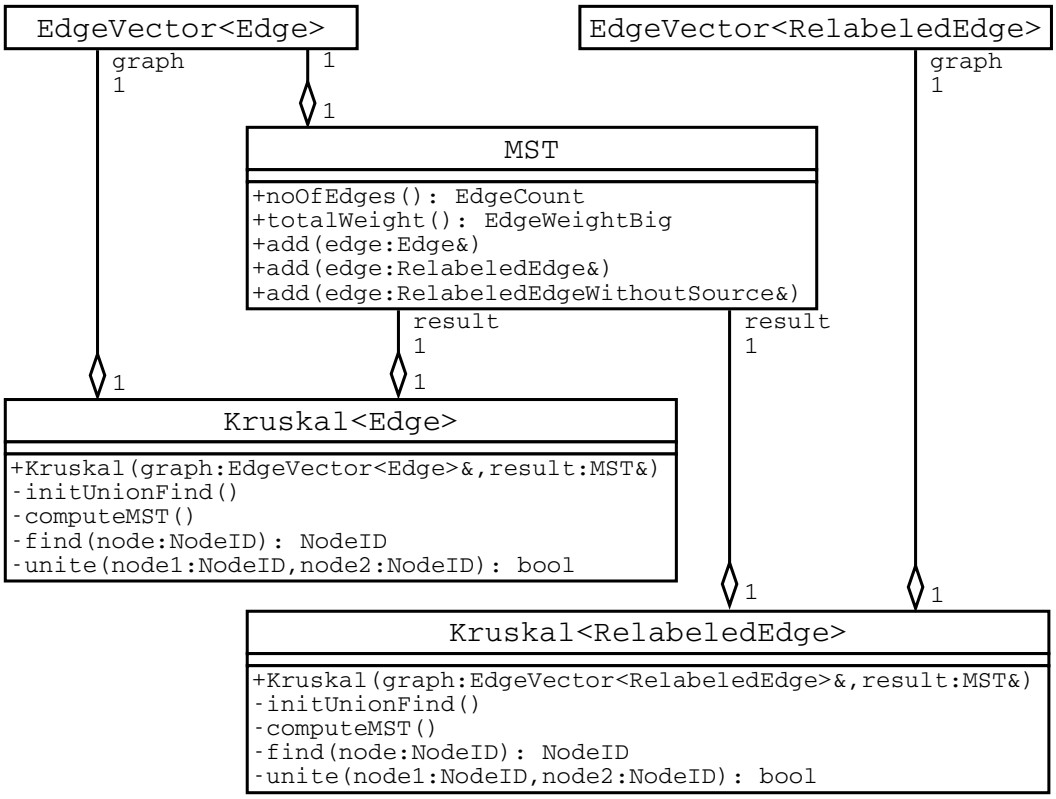


Figure 4.2: UML class diagram — Kruskal's algorithm

first bucket is an exception as we want to use it as input for Kruskal's algorithm that needs a more flexible access since it has to sort the edges. Hence, we use an `EdgeVector` as the first bucket.

The size of the first bucket is the number of nodes that fit in internal memory. The size of the other buckets should be not too small (otherwise too many buckets are needed and the buffers of the buckets exceed the memory limit) and not too large (otherwise the edges of one external bucket do not fit in internal memory). It is convenient to choose the same size for all (but the first) buckets as the computation of the appropriate bucket identifier for a given node is simplified. At first sight this seems not to be the best choice since the buckets with lower IDs probably contain much more edges than the buckets with higher IDs if all buckets have the same size (cp. 3.3.4). But, on the other hand, when the buckets with the higher IDs have been processed, their buffers are not needed any more, and so the released memory can be used to store more edges in internal memory.

4.4.2 Internal Buckets

The internal buckets have to be very flexible as for each node the number of edges can be very different and is not known in advance. Furthermore, the internal buckets are reused several times. For instance, the last internal bucket contains the edges of u_b, then the edges of u_{b-1} and finally the edges of u_1. When it has adapted its size to u_b, it is possible that this size is entirely improper for u_{b-1}.

The usage of one std::vector for each internal bucket would lead to a waste of either memory or time: if the vectors are not reinitialized after the processing of each external bucket, the total capacity increases continuously so that it exceeds the total number of edges by a high factor. On the other hand, the reinitialization takes time.

To avoid these problems, we use a `CommonPoolOfBlocks`, which is shared by all internal buckets. The `CommonPoolOfBlocks` manages a linked list of free blocks. Each block has a small constant capacity to store edges. By invoking the `request`-method a internal bucket can get a pointer to a free

block, which is removed from the free list and can be used exclusively by the requesting internal bucket to store its edges. An internal bucket can give a block back to the pool by calling the `release`-method. Due to these measures the unused capacity is at any time less than the number of internal buckets *times* the capacity of one block because for each internal bucket less than one whole block is unused.

Basically, we need to add edges to internal buckets (when the edges of an external bucket are distributed to the internal buckets) and remove them later (in order to relabel them). The functionality of a stack that uses the `CommonPoolOfBlocks` is encapsulated by the class `SparingStack`. In our case we additionally need a method `determineMinEdge` in order to iterate through all edges to find the shortest one. This method is provided by the subclass `REWS_SparingStack` that is specialized in storing `RelabeledEdgeWithoutSource`-objects. Thus, each internal bucket is represented by one `REWS_SparingStack`.

A `SparingStack` consists of at least one block that does not belong to the `CommonPoolOfBlocks` and therefore is never released. This saves time because the first block does not have to be requested, and usually an internal bucket is not empty so that at least one block is needed.

Figure 4.3 outlines the data structures that are used to implement the internal buckets.

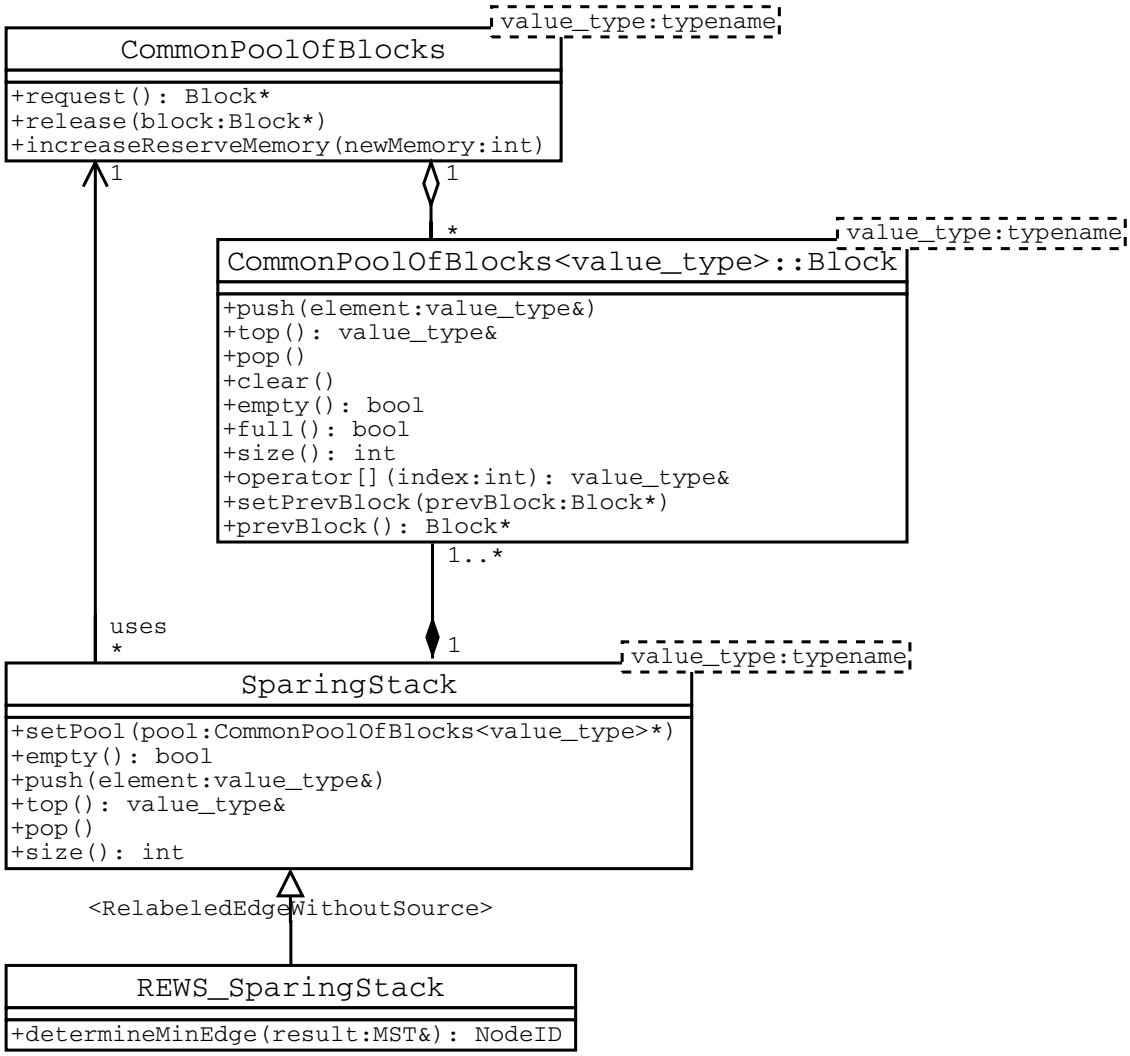


Figure 4.3: UML class diagram — internal buckets

4.4.3 Interface

The class `Buckets` provides an interface for the node reduction. The constructor is given (among others) a reference to a graph represented by an `EdgeVector` and a reference to the `MST`-object that stores the resulting MST. The class `Buckets` aggregates both the external and internal buckets, and it performs the node reduction. After the node reduction has been completed, the method `getIntMemBucket` returns a pointer to the first external bucket that contains the reduced graph (= the nodes that fit in internal memory). Figure 4.4 represents the class `Buckets`.

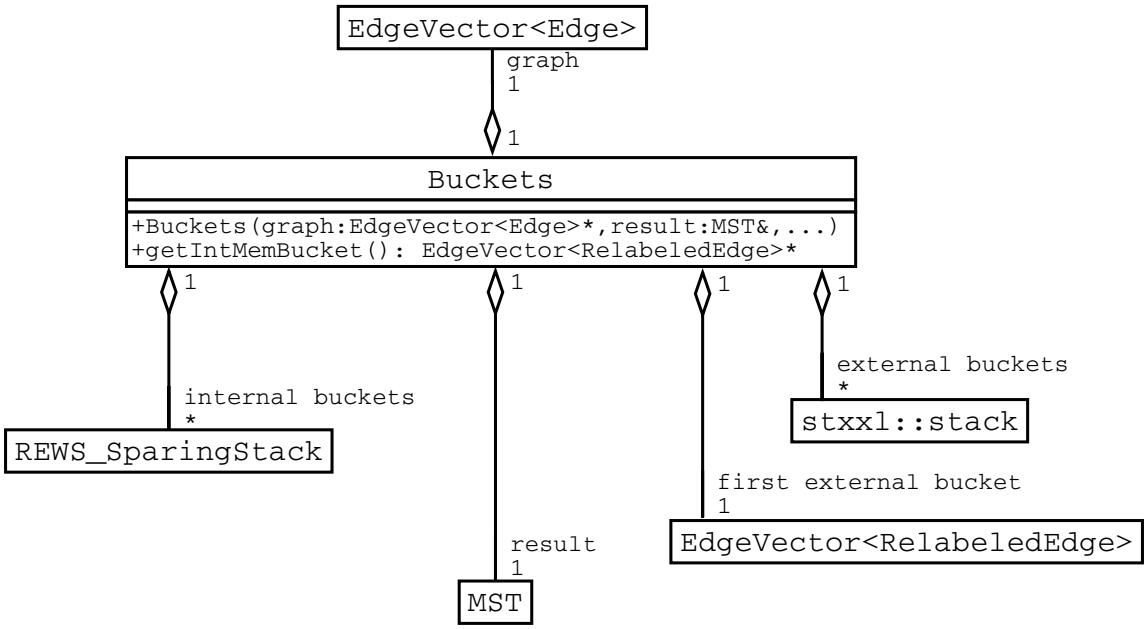


Figure 4.4: UML class diagram — node reduction with buckets

4.5 Node Reduction with a Priority Queue

We use an `EdgeVector` as the first external bucket (cp. 4.4.1) and a `stxxl::priority_queue`. An implementation of the node reduction algorithm presented in Section 3.3.7 is straightforward with the help of these data structures.

The interface of the `PQueue` class, which performs the node reduction and aggregates for this purpose both the first external bucket and the priority queue, is virtually identical with the interface of the `Buckets` class (4.4.3). Therefore, the following sections and figures apply to both the `Buckets` and the `PQueue` implementation, although they refer only to the first one (to simplify matters).

4.6 Main Program

The sequence of the main program is quite simple:

1. import or generate the graph
2. perform the node reduction
3. use the reduced graph as input for Kruskal's algorithm

Figure 4.5 represents this sequence by a diagram. The second step is skipped if the input graph is so small that it can be processed by Kruskal's algorithm immediately.

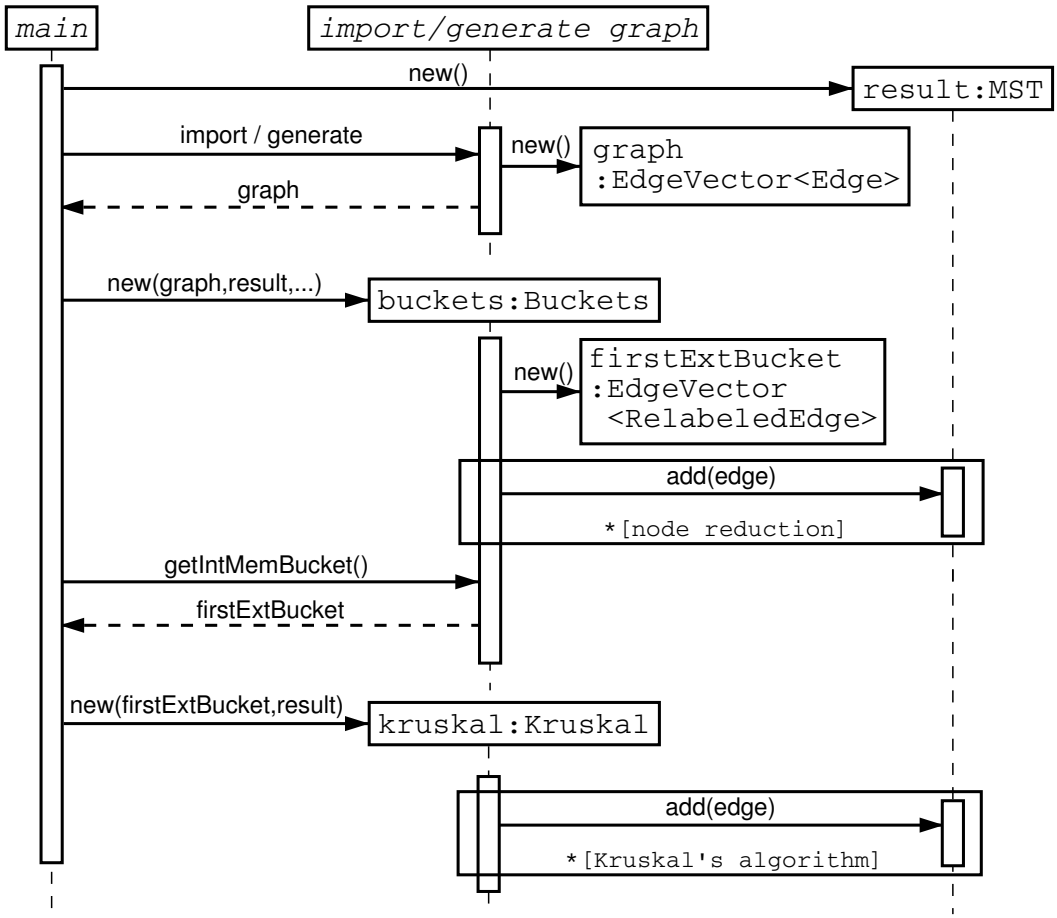


Figure 4.5: UML sequence diagram

4.7 Randomization

The randomization of the node indices is essential for the expected running time (cp. 3.3.4). Hence, we apply a (pseudo-)random permutation on the node indices before the nodes are distributed to the external buckets. As not all node indices fit in internal memory at the same time, we cannot use a standard procedure that swaps random elements. Instead, we apply a bijection on each node index, so each edge can be randomized independently (without looking at other edges, just by a relatively simple computation). Of course we need a special bijection that leads to a (pseudo-)random permutation.

We use a variant of a *Feistel permutation* [NR99]. Let x be the node index that should be randomized. We split x into two parts $a := x \operatorname{div} r$ and $b := x \bmod r$, where $r := \lceil \sqrt{n} \rceil$. During one iteration i we perform the following operation: $a' := b, b' := (a + f_i(b)) \bmod r$, where $f_i(b) \in \{0, \ldots, r-1\}$ is a random number taken from a table that has been computed once. This step is executed twice (we get a'' and b''). Finally, a'' and b'' are recombined to obtain the randomized node index $x' = a'' \cdot r + b''$.

Originally, this is a bijection over $\{0, \ldots, r^2 - 1\}$, but we want a bijection over $\{0, \ldots, n-1\}$[1], so we repeat the application of the bijection, if necessary, until $x' \in \{0, \ldots, n-1\}$.

[1] In Section 3.3 we use node indices from 1 to n for the abstract descriptions. However, node indices from 0 to $n-1$ are more convenient for the implementation.

4.8 Removal of Parallel Edges

By relabeling it is possible that parallel edges are created, i.e., edges that lead from the same source vertex to the same target vertex. As a minimum spanning tree contains at the most the shortest one of several parallel edges, the redundant duplicates can and should be ignored for further processing. Hence, it is reasonable to remove these duplicates. As the removal of parallel edges is not very expensive, the disadvantage of looking for them in graphs that do not have many of them is small; but, on the other hand, the advantage for special graphs, grid graphs for example, is clearly noticeable.

The removal of duplicates is integrated in the relabeling step. Instead of adding the relabeled edges of the last node directly to the appropriate external or internal bucket, they are first added to a hash map by calling the `insert`-method of the `DuplicatesRemover`-class, which aggregates the hash map. If an edge with the same source and the same target vertex is already stored in the hash map, it is replaced with the new edge if the new edge is shorter, otherwise the new edge is discarded. If the capacity of the hash map is exhausted, further edges are written to the appropriate external or internal bucket directly; this limits the waste of time when there are many different edges.

When all edges of the last node have been inserted in the hash map, it has to be cleared and the edges have to be written to the appropriate bucket. In order to be able to clear a hash map fast (especially if it contains only few elements), the inserted elements are additionally stored in an array without gaps. Each entry in the hash map consists of the edge and the index in the array where the edge is additionally stored, so the element in the array can be updated in constant time when the corresponding element in the hash map is replaced by a shorter edge. Due to this data structure the hash map can be cleared by iterating through the array (instead of the whole map) and deleting the elements in the map selectively.

4.9 Ideas for Further Improvements

There are several possible improvements that have not been implemented (yet).

- *Pipelining*. Some I/Os could be saved, if the sort and the scan part of Kruskal's algorithm were combined by a pipeline. Instead of writing the first sorted elements back and reading them later, we could process the sorted elements immediately. Furthermore, we could join the node reduction and the sorting: instead of writing all edges in an unsorted sequence to the first external bucket, we could gather a certain amount of edges in order to build sorted runs. Then we only have to merge these sorted runs. Planned enhancements of the `<stxxl>` library will make such improvements possible.

- *Exception handling in the buckets implementation.* Currently, the buckets implementation cannot deal with every imaginable graph. As mentioned in Section 3.3.7, we get into trouble if the graph has a small average vertex degree, but contains some nodes with a very a high degree. In this case reading the external bucket that contains these exceptional nodes could fail because not all edges of this external bucket fit in the internal buckets. In order to handle this exceptional cases, we could switch temporarily to the priority queue implementation when we realize that the current external bucket would not fit in internal memory.
  This improvement has not been implemented since such degenerated graphs are quite rare and, if necessary, the priority queue implementation could be used right from the start.

- *Adaptive bucket sizes.* In our buckets implementation all external buckets (but the first) have the same size. As described in Section 4.4.1, this has some advantages. However, it could be worthwhile to try using adaptive bucket sizes, i.e., the buckets with lower IDs contain the edges of less nodes as the average vertex degree increases from the last to the first bucket.

- *Intermediate buckets.* When an external bucket is read and when edges are relabeled, they are distributed to random internal buckets. This leads to many cache misses. In order to reduce the number of cache misses, it could be reasonable to install some intermediate buckets between the existing external and internal buckets, so the concept of two layers of data (introduced in Section 3.3.6) would be extended to three layers and memory accesses would be no longer distributed over the whole internal memory.

Chapter 5

Evaluation

5.1 Test Data

As there is a lack of real-world data, we use generated graphs to measure the runtime performance. Three different graph families are examined [MS94][1].

1. random graphs with a given number of vertices and a given number of edges: for each edge a random weight and two random endpoints are selected,

2. grid graphs with $nodesX \cdot nodesY$ vertices: each vertex is connected with its four neighbours (except the marginal nodes, which are connected with three resp. two neighbours), the edges have random weights,

3. geometric graphs: the given number of vertices is placed in a square, each vertex is connected with the given number of nearest neighbours, the distance between two nodes is the square of the Euclidean distance (the extraction of the root is insignificant in respect of the sequence of the algorithm and would slow down the graph generation unnecessarily), parallel edges are removed.

Apart from the grid graphs it is possible that the generated graphs are not connected. Especially (very) sparse random or geometric graphs, which have been generated that way, are almost never connected. We do not take any measures to remedy this unwanted state because, in the main, the sequence of the program is independent of the connectivity of the graph: if a connected graph is given, a minimum spanning *tree* will be determined; if an unconnected graph is given, a minimum spanning *forest* will be found. Actually, the only difference is the chance of an earlier abort if $n - 1$ edges have been added to the resulting MST: if we deal with an unconnected graph, we will never be able to fulfill this abort condition, so we will have to scan through all edges. Hence, the fact that we do not make the generated graphs connected in any case leads at the most to a slight slowdown.

5.2 Test Environment and Settings

The evaluation is done on a machine with two 2GHz Intel Xeon processors, 1 GB RAM and four disks (80 GB each) with a total I/O bandwidth of up to 180 MB/s [DS03]. Debian Linux with kernel version 2.4.20 is used as operating system. The chosen filesystem is XFS and the swap file has been disabled.

Unless otherwise specified, we use the buckets implementation with the following parameters:

- 4 hard disks (and appropriate parameters to take advantage of the parallelism),

[1] Moret and Shapiro additionally use graphs that represent the worst case for Prim's resp. Kruskal's algorithm. As our implementation has nothing to do with Prim's algorithm, we could not expect informative results if we evaluated the former. We do not explicitly use the latter, either, because, in contrast to Moret and Shapiro, we also process unconnected graphs so that the worst case for Kruskal's algorithm (namely that all edges have to be sorted and scanned) occurs anyway.

- 2 MB block size for `stxxl::vectors` (particularly for the first external bucket) and 512 KB block size for `stxxl::stacks` (i.e., for all other external buckets),
- the first external bucket contains the edges of the first 160,000,000 nodes (the union find data structure of these nodes fits in internal memory), the other external buckets contain the edges of 1,800,000 nodes,
- consequently, there are 1,800,000 internal buckets (each of them possesses one block that can store up to 8 edges[2]), initially the common pool, which can be used by all internal buckets, consists of 1,500,000 blocks (8 edges each),
- 650 MB of internal memory are used for sorting,
- randomization and removing of parallel edges are switched on.

5.3 Test Runs and Results

5.3.1 Main Results

Table 5.1 represents the main results, namely the results of test runs with the three graph families, several sizes and densities. If $n \leq$ 160,000,000, we are involved with a semi-external test case, otherwise with an external one. The number of processed edges p and the number of removed parallel edges (duplicates) d refer to the node reduction phase, so these columns are blank in semi-external cases.

Most of these test runs were done with the above mentioned settings, only the external bucket size was decreased for test cases with $m \approx 4 \cdot n$ resp. $m \approx 8 \cdot n^3$ so that all edges of one external bucket fit in internal memory in any case.[4]

The results of the semi-external test runs do not show wide differences. It is not possible to distinguish between the different graph families. The more edges are processed the more time per edge is spent, but this complies with the expected behaviour as the time complexity of Kruskal's algorithm is in $O(m \ln m)$. For example, if you compare the random graph with $10 \cdot 10^6$ nodes and $80 \cdot 10^6$ edges with the random graph with $160 \cdot 10^6$ nodes and $1,280 \cdot 10^6$ edges, the time per edge differs (1.77μs to 2.05μs), but $t/(m \ln m) \approx 97$ns in both cases. The denser the graph the less time per edge is taken. A denser graph with the same number of edges consists of less nodes, so a MST of a denser graph consists of less edges. Hence, there are more find operations with negative results (i.e., both nodes already belong to the same set) so that less union operations are needed and the height of the trees becomes very small due to path compression. Therefore, less time is needed when we deal with denser graphs.

Obviously, the external test runs are slower than the semi-external ones, but fortunately the differences keep within reasonable limits. When we look at grid resp. geometric graphs, the differences even decrease when the graph size increases. Mainly, this effect is due to the removal of parallel edges. The more edges in a grid or geometric graph the greater the rate of removed edges (cp. column d/m). Hence, the number of edges that have to be processed by Kruskal's algorithm is kept small. For example, $5.6 \cdot 10^8$ edges of a grid graph with $640 \cdot 10^6$ nodes survive the node reduction, and $6.3 \cdot 10^8$ edges of a graph that is twice this size. Furthermore, the removal of duplicates is one of two reasons why the number of processed edges is distinctly less than the expected number of processed edges (cp. column $p/E(p)$) when we deal with large instances. The other reason is the fact that the analysis of the time complexity (Section 3.3.4) is rather cautious. For instance, it is not regarded that for each node at least one edge, which is added to the MST, is eliminated. Unfortunately, only the second reason applies to random graphs as the removal of parallel edges is not effective. (There are some multiple edges, but distinctly less than 1%.) Hence, the number of

[2] The more edges in one block the greater the extent of unused capacity (cp. 4.4.2) and the smaller the temporal overhead of operations on linked lists of blocks — and vice versa. Hence, 8 edges is a compromise.

[3] Originally, we wanted to evaluate test cases with $m = 2 \cdot n$, $m = 4 \cdot n$ and $m = 8 \cdot n$, but we had to restrict ourselves to approximate values as the average vertex degree of a grid graph is slightly less than four and the average vertex degree of a geometric graph depends on the given number of nearest neighbours and cannot be set to an exact value.

[4] Furthermore, the size of the first external bucket was reduced to 150,000,000 for large geometric graphs due to a slight misfeature of the memory management caused by the expensive geometric graph generator.

type	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$	$p/10^6$	$p/E(p)$	d/m
grid	40	80	177	2.21			
grid	80	160	362	2.27			
grid	160	320	738	2.31			
grid	320	640	2 535	3.96	750	85 %	4 %
grid	640	1 280	4 712	3.68	2 492	70 %	13 %
grid	1 280	2 560	9 056	3.54	6 167	58 %	22 %
random	40	80	185	2.32			
random	80	160	388	2.42			
random	160	320	813	2.54			
random	320	640	2 773	4.33	766	86 %	0 %
random	640	1 280	6 098	4.76	2 752	78 %	0 %
random	1 280	2 560	14 202	5.55	7 676	72 %	0 %
random	20	80	155	1.94			
random	40	160	318	1.99			
random	80	320	676	2.11			
random	160	640	1 427	2.23			
random	320	1 280	5 889	4.60	1 651	93 %	0 %
random	640	2 560	14 248	5.57	6 284	89 %	0 %
random	10	80	142	1.77			
random	20	160	286	1.79			
random	40	320	591	1.85			
random	80	640	1 242	1.94			
random	160	1 280	2 627	2.05			
random	320	2 560	12 370	4.83	3 426	97 %	0 %
geometric	40	75	183	2.45			
geometric	80	149	377	2.53			
geometric	160	298	787	2.64			
geometric	320	596	2 175	3.65	644	78 %	7 %
geometric	640	1 190	3 797	3.18	1 949	59 %	13 %
geometric	1 280	2 390	7 278	3.05	4 575	45 %	15 %
geometric	20	71	148	2.09			
geometric	40	141	300	2.13			
geometric	80	282	627	2.22			
geometric	160	564	1 333	2.36			
geometric	320	1 130	4 126	3.66	1 275	82 %	18 %
geometric	640	2 260	7 004	3.10	3 975	61 %	34 %
geometric	10	68	124	1.84			
geometric	20	135	246	1.82			
geometric	40	270	511	1.89			
geometric	80	540	1 067	1.98			
geometric	160	1 080	2 209	2.04			
geometric	320	2 160	7 549	3.49	2 650	81 %	30 %

n nodes, m edges, t elapsed time, p processed edges, $E(p)$ expected value of p according to 3.3.4, d duplicates (parallel edges) removed

Table 5.1: (Semi-)External test cases

processed edges is less than the expected number, but greater than the corresponding number at test runs with grid resp. geometric graphs. Therefore, the time per edge increases when we deal with larger random graphs. This is the "normal" behaviour as the time complexity of the node reduction algorithm is not in $O(m)$. If you regard the time per processed edge, you can find out that this quantity even decreases when the graph size increases.

At first sight it is surprising that test runs with denser graphs are partly as slow as test runs with sparser graphs. For instance, the time per edge for a random graph with $2,560 \cdot 10^6$ edges and $m = 2 \cdot n$ and for a random graph with the same number of edges, but $m = 4 \cdot n$, is almost identical (about 5.5μs each). As the number of processed edges depends not only on m, but also on n, we would have expected that the test run with $m = 4 \cdot n$ is faster. However, the sparser graphs benefit from another fact: due to the larger number of nodes the node reduction phase actually takes a longer time (10,571s instead of 9,177s, for the above mentioned example), but, on the other hand, more edges are eliminated because for each node at least the shortest edge is removed, so there are less edges that have to be processed by Kruskal's algorithm (about $1.5 \cdot 10^9$ instead of $2.1 \cdot 10^9$). Hence, Kruskal's algorithm is faster (3,631s instead of 5,071s) and compensates for the slower node reduction phase.

5.3.2 Comparison with Internal Implementations

In order to be able to judge the performance of our implementation, we need comparison values. Therefore, we fall back on internal implementations of Kruskal's and of Prim's algorithm developed at the Max-Planck-Institute for Computer Science by Irit Katriel. We used random graphs generated by Irit's program and grid and geometric graphs generated by our program. As the implementation of Prim's algorithm requires more memory, some instances were processed only by Kruskal'a algorithm. Table 5.2 contains the results of the internal test runs.

			Kruskal		Prim	
type	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$	$t[s]$	$t/m[\mu s]$
grid	2.5	5.0	7.5	1.50	3.8	0.75
grid	5.0	10.0	15.2	1.52	8.2	0.82
random	2.5	5.0	6.5	1.30	10.0	1.99
random	5.0	10.0	13.5	1.35	22.1	2.21
random	10.0	20.0	28.2	1.41		
random	1.3	5.0	5.3	1.07	6.1	1.22
random	2.5	10.0	10.9	1.09	12.9	1.29
random	5.0	20.0	22.4	1.12		
random	0.6	5.0	4.7	0.94	3.7	0.73
random	1.3	10.0	9.6	0.96	7.5	0.75
random	2.5	20.0	19.9	1.00		
geometric	2.5	4.7	7.3	1.56	5.6	1.19
geometric	5.0	9.3	14.5	1.56	13.1	1.41
geometric	1.3	4.4	6.3	1.42	2.9	0.66
geometric	2.5	8.8	12.6	1.43	6.4	0.73
geometric	0.6	4.2	5.3	1.26	1.7	0.41
geometric	1.3	8.4	10.8	1.27	3.6	0.43

Table 5.2: Internal test cases

In the main, both implementations show the expected behaviour. Kruskal's algorithm is quite independent of the graph type. When denser graphs are processed, Kruskal's algorithm gets faster due to the same reasons that applied to the semi-external test cases described in Section 5.3.1. One basic feature of Prim's algorithm is the fact that the time per edge decreases when the density increases. This feature is confirmed by our results. Furthermore, Prim's algorithm is more efficient when grid or geometric graphs are processed.

In Table 5.3, we compare external test runs with internal ones. For each graph type and for each density, the last entry in Table 5.1 is compared with the last entry in Table 5.2. The time per edge of the external test case is divided by the time per edge of the corresponding internal test case for both algorithms, Kruskal's and Prim's.

		$(t/m)_{\text{ext}} : (t/m)_{\text{int}}$	
type	*density*	Kruskal	Prim
grid	$m \approx 2 \cdot n$	2.3	4.3
random	$m \approx 2 \cdot n$	3.9	2.5
random	$m \approx 4 \cdot n$	5.0	4.3
random	$m \approx 8 \cdot n$	4.8	6.4
geometric	$m \approx 2 \cdot n$	2.0	2.2
geometric	$m \approx 4 \cdot n$	2.2	4.2
geometric	$m \approx 8 \cdot n$	2.7	8.1

Table 5.3: Comparison between external and internal test cases

With regard to the internal implementation of Kruskal's algorithm, our external implementation is between two and five times slower. When we compare our implementation with Prim's algorithm, the factor ranges between 2.2 and 8.1. When we make the analogous comparison between the semi-external version of Kruskal's algorithm and the internal one, we obtain a factor between 1.5 and 2.[5] However, we have to consider that these comparisons are *dis*advantageous to our implementation as we cannot expect that the expense grows only linearly.

Figure 5.1 illustrates the results of internal, semi-external and external test runs with $m \approx 2 \cdot n$. Analogically, the Figures 5.2 and 5.3 show the results of the test runs with $m \approx 4 \cdot n$ resp. $m \approx 8 \cdot n$. To keep the figures easy to survey, we omit the internal test runs with grid and geometric graphs.

5.3.3 Randomization and Removal of Parallel Edges

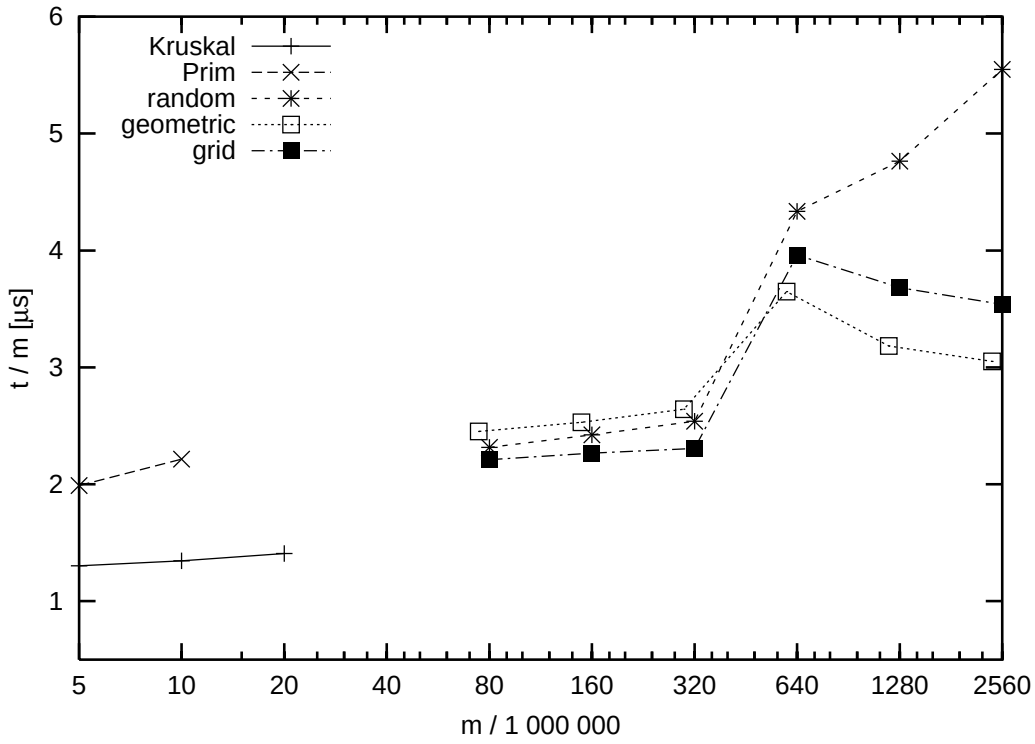
As we wanted to evaluate the benefit of the removal of parallel edges, we reran the external test cases of grid graphs with *de*activated `DuplicatesRemover` (cp. 4.8). Table 5.4 shows both the results with activated and with deactivated `DuplicatesRemover`.

parallel edges	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$	$p/10^6$	$p/E(p)$	d/m
removed	320	640	2 535	3.96	750	85 %	4 %
removed	640	1 280	4 712	3.68	2 492	70 %	13 %
removed	1 280	2 560	9 056	3.54	6 167	58 %	22 %
not removed	320	640	2 539	3.97	760	86 %	
not removed	640	1 280	5 006	3.91	2 642	74 %	
not removed	1 280	2 560	10 171	3.97	6 969	65 %	

Table 5.4: Grid graphs — removal of parallel edges

From these results, we can conclude that the removal of parallel edges becomes worthwhile when the graph size increases. For instance, the `DuplicatesRemover` eliminates 22% ($\approx 5.7 \cdot 10^8$) of all edges from a grid graph with $1.28 \cdot 10^9$ nodes and about $2.56 \cdot 10^9$ edges. When these edges are not removed, most of them are processed more than once, so the number of processed edges even increases from $6.2 \cdot 10^9$ to $7.0 \cdot 10^9$. Hence, the test run with activated `DuplicatesRemover` is more than 10% faster. Furthermore, less internal memory is allocated, so the external bucket size could be increased.

[5] Both the internal and the semi-external algorithm have a number of opportunities for further tuning. Currently, the external sorter benefits from the fact that only integer keys are used, while the internal sorter is comparison based. Hence, bucket sort could accelerate the internal sorter. On the other hand, the external sorter is not optimized for small elements. Furthermore, pipelining (cp. 4.9) has not been implemented, yet. But none of these measures is likely to yield more than a factor of 2.



Kruskal and *Prim* denote the internal test runs with random graphs, *random*, *geometric* and *grid* label the (semi-)external test runs with the corresponding graph type.

Figure 5.1: $m \approx 2 \cdot n$

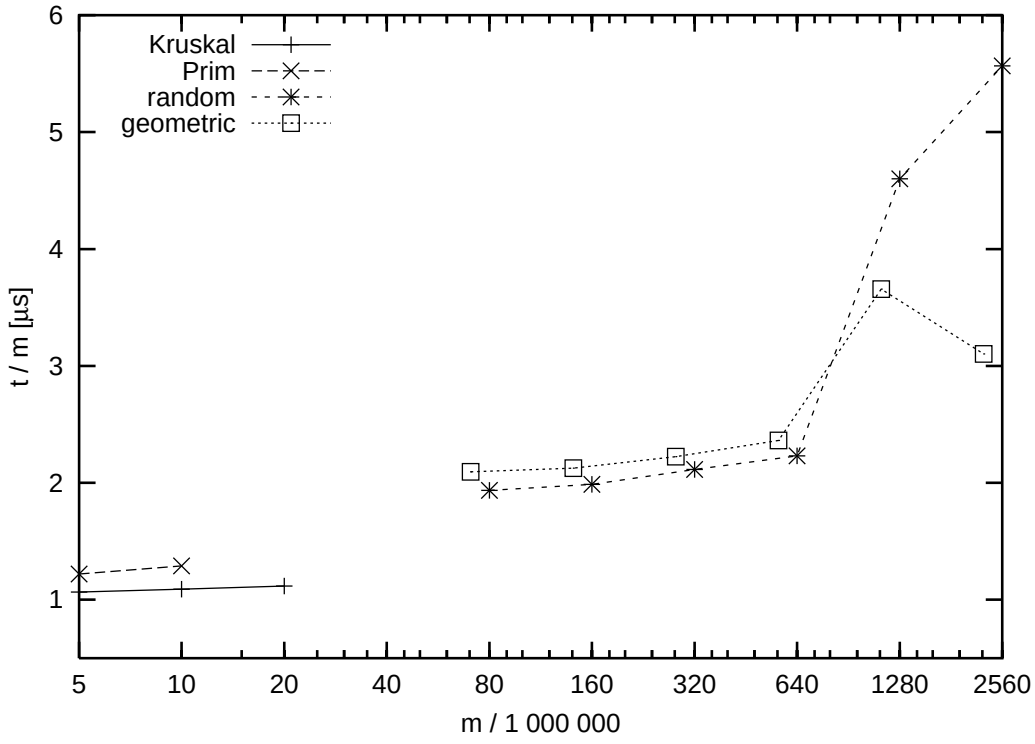


Figure 5.2: $m \approx 4 \cdot n$

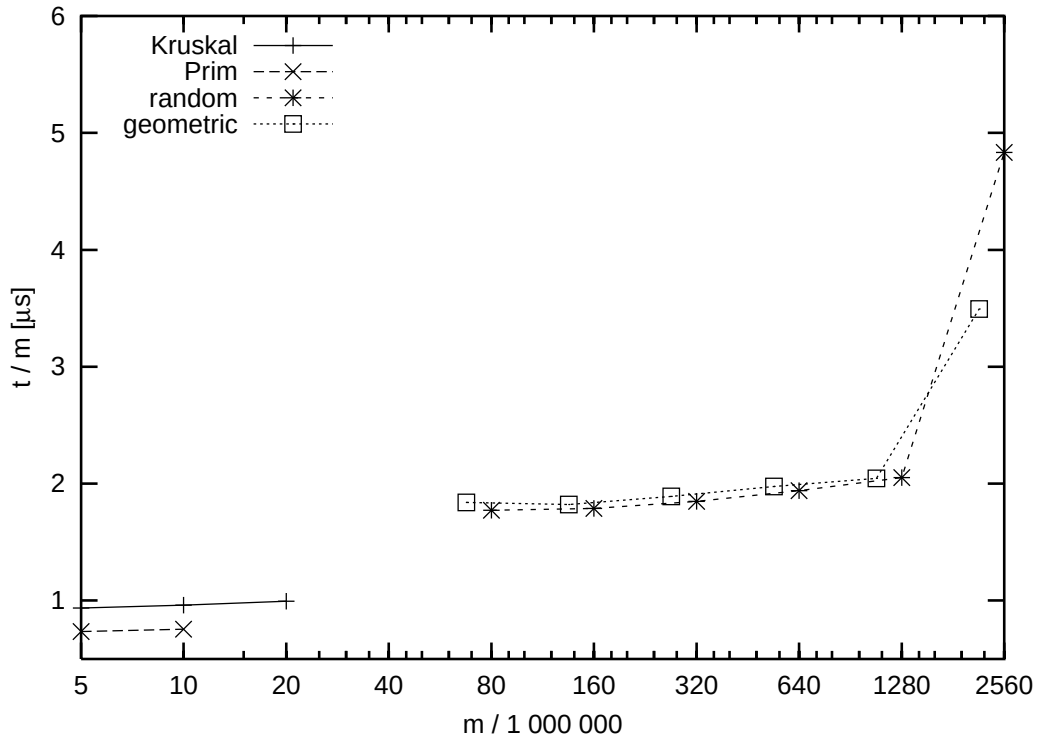


Figure 5.3: $m \approx 8 \cdot n$

The randomization of a random graph is redundant as the node indices are uniformly distributed anyway. Furthermore, the removal of duplicates is not worthwhile since a random graph contains only few parallel edges. Hence, we wanted to find out the extent of the overhead. Table 5.5 represents the results of two test runs with a random graph, one with randomization and removal of parallel edges and one without these measures.

parallel edges	*randomization*	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$	$p/10^6$	$p/E(p)$	d
removed	activated	1 280	2 560	14 202	5.55	7 676	72 %	555
not removed	*de*activated	1 280	2 560	12 465	4.87	7 675	72 %	

Table 5.5: Random graph — randomization and removal of parallel edges

Firstly, these results confirm our conjecture that the randomization of a random graph is superfluous. The number of processed edges is almost identical. Secondly, it is obvious that the removal of duplicates is not worthwhile because only 555 ($\approx 0.00002\%$) parallel edges are eliminated. Finally, the test run without randomization and without `DuplicatesRemover` is about 12% faster. As both measures do not speed up the processing, this difference exactly reflects the expense of the randomization and the removal of parallel edges.

5.3.4 Buckets vs. Priority Queue

In order to compare both implementations, we selected three representative instances and applied both versions one after the other. Table 5.6 shows the different execution times.

The results demonstrate that currently the buckets implementation needs less than half the time of the priority queue implementation. There are two reasons for this. Firstly, the buckets implementation is optimized for the MST problem, while the priority queue of the `<stxxl>` library is a very general data structure. Secondly, the priority queue is not fully developed yet.

implementation	*type*	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$
buckets	grid	320	640	2 535	3.96
priority queue	grid	320	640	6 156	9.62
buckets	random	320	640	2 773	4.33
priority queue	random	320	640	6 013	9.40
buckets	random	1 280	2 560	14 202	5.55
priority queue	random	1 280	2 560	54 497	21.29

Table 5.6: Buckets vs. priority queue

5.3.5 Large Instances

To sound the limits of the program, we processed grid graphs with 2^{31} and with 2^{32} nodes. As the number of external buckets increased, the block size for `stxxl::stacks` had to be reduced.[6] Table 5.7 shows the external test cases of grid graphs including the above mentioned large instances.

block size	$n/10^6$	$m/10^6$	$t[s]$	$t/m[\mu s]$	$p/10^6$	$p/E(p)$	d/m
512 KB	320	640	2 535	3.96	750	85 %	4 %
512 KB	640	1 280	4 712	3.68	2 492	70 %	13 %
512 KB	1 280	2 560	9 056	3.54	6 167	58 %	22 %
256 KB	2 150	4 290	15 803	3.68	11 230	50 %	27 %
128 KB	4 290	8 590	31 081	3.62	26 260	46 %	29 %

Table 5.7: Grid graphs — large instances

Although the block size is halved twice, the time per edge is almost constant and does not increase when very large graphs are processed. The reduced block size is compensated by the `DuplicatesRemover`, which is very efficient for large instances. For example, the number of processed edges is less than half the expected number if we regard the grid graph with 2^{32} nodes. This is achieved by removing 29% of all edges.

[6] Furthermore, the sizes of the external buckets were adapted for the last test run so that they were particularly appropriate for a large grid graph: the first external bucket contained the edges of 150,000,000 nodes (instead of 160,000,000) and all other external buckets contained the edges of 2,500,000 nodes (instead of 1,800,000).

Acknowledgements

First of all, I like to thank my supervisor, Peter Sanders, for the numerous fertile discussions. His suggestions and his optimism were very helpful.

Roman Dementiev always provided an up-to-date version of the `<stxxl>`-library and made sure that the test environment worked. Furthermore, he enhanced the library according to my requirements and supported my hunt for bugs (particularly memory leaks) with great patience.

Job Sibeyn kindly made his not yet published external memory MST algorithm available. Irit Katriel provided internal implementations of Prim's and of Kruskal's algorithm so that I was able to do comparative measurements.

Appendix A

Reference Manual

A.1 Hierarchical Index

A.1.1 Class Hierarchy

This inheritance list is sorted roughly, but not completely, alphabetically:

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CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks > . . . 29
CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block . . . 30
DuplicatesRemover< SuperContainer > . . . 31
EdgeWithoutSource . . . 37
Edge . . . 34
RelabeledEdge . . . 48
RelabeledEdgeWithoutSource . . . 49
GetWeight< EdgeType > . . . 39
Kruskal< EdgeType > . . . 39
MST . . . 42
PQueue . . . 43
Randomizer< Bijection > . . . 45
RandomizerFeistel . . . 46
RandomizerLinearCongruence . . . 47
result
EdgeVector< EdgeType, blockSize, noOfPages, pageSize > . . . 36
SourceTargetWeightOrdering . . . 51
SourceWeightOrdering< EdgeType > . . . 52
SparingStack< value_type, elementsPerBlock, noOfBlocks > . . . 53
SparingStack< RelabeledEdgeWithoutSource, elementsPerBlock, noOfBlocks > . . . 53
REWS_SparingStack< elementsPerBlock, noOfBlocks > . . . 51
WeightOrdering< EdgeType > . . . 54

A.2 Compound Index

A.2.1 Compound List

Here are the classes, structs, unions and interfaces with brief descriptions:

Buckets< **nodesPerBucket** > (Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures) 25
CommonPoolOfBlocks< **value_type, elementsPerBlock, noOfBlocks** > (A CommonPoolOfBlocks manages blocks which can contain several elements of a specified value_type) . 29
CommonPoolOfBlocks< **value_type, elementsPerBlock, noOfBlocks** >**::Block** (A block which can contain several elements) 30
DuplicatesRemover< **SuperContainer** > (A container which can be used as an 'intermediate station' in order to remove multiple edges) 31
Edge (Represents a directed and weighted edge) 34
EdgeVector< **EdgeType, blockSize, noOfPages, pageSize** > (Represents a weighted graph by a list of Edges) 36
EdgeWithoutSource (Represents an edge without the source vertex) 37
GetWeight< **EdgeType** > (Extracts the weight of an edge) 39
Kruskal< **EdgeType** > (Represents Kruskal's algorithm for determining a Minimum Spanning Tree of a weighted graph and the required union/find data structure) 39
MST (Represents a Minimum Spanning Tree (MST) of a graph) 42
PQueue (Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures) 43
Randomizer< **Bijection** > (A bijection over {0,...,n-1} that can be used to randomize an **Edge** (p. 34) in order to obtain a (pseudo-)random permutation) 45
RandomizerFeistel (A bijection that uses Feistel permutations) 46
RandomizerLinearCongruence (A bijection that uses the linear congruential method) 47
RelabeledEdge (Represents a relabeled edge) 48
RelabeledEdgeWithoutSource (Represents a relabeled edge without the source vertex) 49
REWS_SparingStack< **elementsPerBlock, noOfBlocks** > (A specialized **SparingStack** (p. 53) which stores **RelabeledEdgeWithoutSource** (p. 49)-objects) 51
SourceTargetWeightOrdering (A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by the target vertex and finally by weight) 51
SourceWeightOrdering< **EdgeType** > (A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by weight) 52
SparingStack< **value_type, elementsPerBlock, noOfBlocks** > (A stack which stores its elements in blocks which are managed by a **CommonPoolOfBlocks** (p. 29)) 53
WeightOrdering< **EdgeType** > (A StrictWeakOrdering predicate which can be used to compare two edges by weight) 54

A.3 Class Documentation

A.3.1 Buckets< nodesPerBucket > Class Template Reference

Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures.

`#include <buckets.h>`

Public Types

- typedef int **BucketID**

Public Methods

- **Buckets** (**EdgeVector**< **Edge** > &graph, **MST** &result, BucketID noOfBuckets, NodeCount noOfNodesInIntMem)

  Reduces the number of nodes of a graph (in order to compute a minimum spanning tree).

- InternalMemoryBucket * **getIntMemBucket** ()

  Returns a pointer to the first external bucket which contains the edges of the nodes which fit in internal memory.

- void **add** (const **RelabeledEdge** &edge)

  Adds the given edge to the appropriate external bucket or to the appropriate internal bucket if the edge belongs to the external bucket which is processed at the moment.

Static Public Methods

- BucketID **noOfExtBuckets** (NodeCount noOfNodes, NodeCount noOfNodesInIntMem)

  Computes the number of external buckets that are needed.

Private Types

- typedef stxxl::STACK_GENERATOR< **RelabeledEdge**, stxxl::external, stxxl::grow_shrink2, DS_EXT_PAGE_SIZE, DS_EXT_BLOCK_SIZE >::result **EdgesOfSeveralNodes**

  The type of an external bucket which contains the edges of several nodes.

- typedef **REWS_SparingStack**< DS_INT_EDGES_PER_BLOCK, DS_INT_NO_OF_BLOCKS > **EdgesOfOneNode**

  The type of an internal bucket which contains the edges of one node.

- typedef **CommonPoolOfBlocks**< **RelabeledEdgeWithoutSource**, DS_INT_EDGES_PER_BLOCK, DS_INT_NO_OF_BLOCKS > **PoolEdgesOfOneNode**

  The type of the common pool of the internal buckets.

Private Methods

- void **initBuckets** ()

  Initializes the external buckets and distributes the edges to the external buckets.

- void **reduceNodes** ()

  Reduces the number of nodes.

- BucketID **bucketID** (NodeID nodeID) const

  Returns the ID of the bucket which contains the edges of the given node.

- void **addToExternalBucket** (const **RelabeledEdge** &edge)

  Adds the given edge to the appropriate external bucket.

- void **addToExternalBucket** (const **RelabeledEdge** &edge, BucketID newBucketID)

  Adds the given edge to the bucket which is specified by newBucketID.

Private Attributes

- **EdgeVector**< **Edge** > & **_graph**

  Reference to the given graph.

- **MST** & **_result**

  Reference to a **MST** (p. 42) *object which stores the result.*

- NodeCount **_noOfNodesInIntMem**

  The number of nodes which fit in internal memory.

- InternalMemoryBucket * **_firstExtBucket**

  The first external bucket which contains the edges of the nodes which fit in internal memory.

- std::vector< **EdgesOfSeveralNodes** * > **_extBuckets**

  The external buckets.

- stxxl::prefetch_pool< EdgesOfSeveralNodes::block_type > **_prefetchPool**

  The prefetch pool that is used by the external buckets.

- stxxl::write_pool< EdgesOfSeveralNodes::block_type > **_writePool**

  The write pool that is used by the external buckets.

- **EdgesOfOneNode _intBuckets** [nodesPerBucket]

  The internal buckets.

- NodeID **_firstNodeIDofCurrentBucket**

  The identifier of the first node in the current external bucket will be used to compute the internal bucket index of a node.

A.3.1.1 Detailed Description

template<NodeCount nodesPerBucket = 1> class Buckets< nodesPerBucket >

Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures.

This implementation uses several buckets.

template parameter: The number of nodes per bucket. This value doesn't apply to the first bucket, because it contains the edges of the first "_noOfNodesInIntMem" nodes.

A.3.1.2 Constructor & Destructor Documentation

template<NodeCount nodesPerBucket = 1> Buckets< nodesPerBucket >::Buckets (EdgeVector< Edge > & *graph*, MST & *result*, BucketID *noOfBuckets*, NodeCount *noOfNodesInIntMem*) `[inline]`

Reduces the number of nodes of a graph (in order to compute a minimum spanning tree).

Parameters:

graph a reference to an **EdgeVector** (p. 36) which represents the graph

result a reference to a **MST** (p. 42) object which stores the result

noOfBuckets the number of external buckets which should be used

noOfNodesInIntMem the number of nodes which fit in internal memory

A.3.1.3 Member Function Documentation

template<NodeCount nodesPerBucket> Buckets< nodesPerBucket >::BucketID Buckets< nodesPerBucket >::bucketID (NodeID *nodeID*) const `[inline, private]`

Returns the ID of the bucket which contains the edges of the given node.

The first external bucket has the ID -1 because it is a special case. The second bucket is the first element of _extBuckets and has the ID 0 and so on.

template<NodeCount nodesPerBucket> void Buckets< nodesPerBucket >::reduceNodes () `[private]`

Reduces the number of nodes.

Only the first bucket "survives".

A.3.1.4 Member Data Documentation

template<NodeCount nodesPerBucket = 1> std::vector<EdgesOfSeveralNodes*> Buckets< nodesPerBucket >::_extBuckets `[private]`

The external buckets.

Each bucket contains the edges of several nodes.

template<NodeCount nodesPerBucket = 1> EdgesOfOneNode Buckets< nodesPerBucket >::_intBuckets[nodesPerBucket] `[private]`

The internal buckets.

Each bucket contains the edges of one node.

The documentation for this class was generated from the following files:

- buckets.h
- buckets.cpp

A.3.2 CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks > Class Template Reference

A CommonPoolOfBlocks manages blocks which can contain several elements of a specified value_type.

`#include <sparingStack.h>`

Public Methods

- **CommonPoolOfBlocks** ()
  The default constructor.
- **~CommonPoolOfBlocks** ()
  The destructor.
- **Block * request** ()
  Returns a pointer to a free block.
- void **release** (**Block** *block)
  Adds a block which is no longer used to the free-list.
- void **increaseReserveMemory** (int newMemory)
  Informs that more reserve memory is available.

A.3.2.1 Detailed Description

template<typename value_type, int elementsPerBlock, int noOfBlocks> class CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >

A CommonPoolOfBlocks manages blocks which can contain several elements of a specified value_type.

It can be used by several SparingStacks.

A.3.2.2 Constructor & Destructor Documentation

template<typename value_type, int elementsPerBlock, int noOfBlocks> CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::~CommonPoolOfBlocks () `[inline]`

The destructor.

Blocks which have been created additionally are deleted.

A.3.2.3 Member Function Documentation

template<typename value_type, int elementsPerBlock, int noOfBlocks> void CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::increaseReserveMemory (int *newMemory*) `[inline]`

Informs that more reserve memory is available.

Parameters:

newMemory the space in bytes which can be used by the request method if more blocks are required.

The documentation for this class was generated from the following file:

- sparingStack.h

A.3.3 CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block Class Reference

A block which can contain several elements.

`#include <sparingStack.h>`

Public Methods

- **Block** ()

  The default constructor.

- void **push** (const value_type &element)

  Adds a element to the block.

- const value_type & **top** () const

  Returns a reference to the last element.

- void **pop** ()

  Removes the last element.

- void **clear** ()

  Removes all elements so that the block is empty.

- bool **empty** () const

  Returns true iff the block contains no elements.

- bool **full** () const

  Returns true iff the block can't adopt more elements.

- int **size** () const

  Returns the number of elements which are stored in the block.

- const value_type & **operator[ ]** (int index) const

  Returns a reference to the element which is stored at the given position.

- void **setPrevBlock** (Block *const prev)

  Sets the pointer to the previous block.

- Block * **prevBlock** () const

  Returns a pointer to the previous block.

Private Attributes

- int **_size**

  The number of elements which are stored in the block.

- Block * **_prev**

  A pointer to the previous block.

- value_type **_elements** [elementsPerBlock]

  The elements.

A.3.3.1 Detailed Description

template<typename value_type, int elementsPerBlock, int noOfBlocks> class CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block

A block which can contain several elements.

A.3.3.2 Member Function Documentation

template<typename value_type, int elementsPerBlock, int noOfBlocks> void CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block::pop () `[inline]`

Removes the last element.

Precondition(!): **empty**() (p. 30) must return false.

template<typename value_type, int elementsPerBlock, int noOfBlocks> void CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block::push (const value_type & *element*) `[inline]`

Adds a element to the block.

Precondition(!): **full**() (p. 30) must return false.

template<typename value_type, int elementsPerBlock, int noOfBlocks> const value_type& CommonPoolOfBlocks< value_type, elementsPerBlock, noOfBlocks >::Block::top () const `[inline]`

Returns a reference to the last element.

Precondition(!): **empty**() (p. 30) must return false.

The documentation for this class was generated from the following file:

- sparingStack.h

A.3.4 DuplicatesRemover< SuperContainer > Class Template Reference

A container which can be used as an 'intermediate station' in order to remove multiple edges.

`#include <duplicatesRemover.h>`

Public Methods

- **DuplicatesRemover** (SuperContainer *superContainer)

  Creates a DuplicatesRemover.

- void **insert** (const **RelabeledEdgeWithoutSource** &edge, NodeID source)

  Adds an edge to the container.

- void **clear** (NodeID source)

  Clears the container.

Private Types

- typedef std::pair< **RelabeledEdgeWithoutSource**, int > **HashMapElement**

  The type of a hashMap-element.

Private Methods

- int **hashFunction** (NodeID x) const

  Returns the hash value for the given node ID.

- int **find** (NodeID x) const

  Returns the position of an edge with the given node ID in the hashMap.

- void **clearLocal** (NodeID x)

  Removes entries from the hashMap beginning with the hash value of the given node ID and ending with the first empty entry which is found.

- bool **empty** (int index) const

  Returns true iff the given position in the hashMap is empty.

Private Attributes

- **RelabeledEdgeWithoutSource _edges** [**_maxSize**]

  This array contains all edges without vacancies.

- **HashMapElement _hashMap** [**_hashMapSize**]

  The hashMap which contains the edges.

- EdgeCount **_size**

  The number of edges which are stored in this container.

- SuperContainer * **_superContainer**

  A pointer to the super container, i.e.

Static Private Attributes

- const EdgeCount **_maxSize** = 1024

  The capacity of the container.

- const EdgeCount **_hashMapSize** = 2 ∗ **_maxSize**

  The size of the hash map.

A.3.4.1 Detailed Description

template<typename SuperContainer> class DuplicatesRemover< SuperContainer >

A container which can be used as an 'intermediate station' in order to remove multiple edges.

If there is more than one edge with the same target node (the source node is always the same during one pass), only the edge with minimum weight is preserved. If the capacity of the container is exhausted, further edges which can't be stored are output directly.

A.3.4.2 Member Typedef Documentation

template<typename SuperContainer> typedef std::pair<RelabeledEdgeWithoutSource, int> DuplicatesRemover< SuperContainer >::HashMapElement `[private]`

The type of a hashMap-element.

The first component stores the edge, the second the index in the array '_edges'.

A.3.4.3 Constructor & Destructor Documentation

template<typename SuperContainer> DuplicatesRemover< SuperContainer >::DuplicatesRemover (SuperContainer ∗ *superContainer*) `[inline]`

Creates a DuplicatesRemover.

Parameters:

superContainer a pointer to the super container, i.e. the container that uses this DuplicatesRemover (either a **Buckets** (p. 25) or a **PQueue** (p. 43) object). This is required in order to be able to put the edges to the super container when the capacity of this container is exhausted or when this container is cleared.

A.3.4.4 Member Function Documentation

template<typename SuperContainer> void DuplicatesRemover< SuperContainer >::clear (NodeID *source*) `[inline]`

Clears the container.

The edges are written to the appropriate buckets.

template<typename SuperContainer> int DuplicatesRemover< SuperContainer >::find (NodeID *x*) const `[inline, private]`

Returns the position of an edge with the given node ID in the hashMap.

If no appropriate edge is found, the position where such an edge should be stored is returned.

template<typename SuperContainer> void DuplicatesRemover< SuperContainer >::insert (const RelabeledEdgeWithoutSource & *edge*, NodeID *source*) `[inline]`

Adds an edge to the container.

If an edge with the same target node is already in the container, the new edge replaces the old one if it has a lower weight, otherwise it is discarded.

A.3.4.5 Member Data Documentation

template<typename SuperContainer> RelabeledEdgeWithoutSource DuplicatesRemover< SuperContainer >::_edges[_maxSize] `[private]`

This array contains all edges without vacancies.

This is useful in order to clear the container without traversing the whole hashMap.

template<typename SuperContainer> SuperContainer* DuplicatesRemover< SuperContainer >::_superContainer `[private]`

A pointer to the super container, i.e.

the container that uses this DuplicatesRemover (either a **Buckets** (p. 25) or a **PQueue** (p. 43) object). This is required in order to be able to put the edges to the super container when the capacity of this container is exhausted or when this container is cleared.

The documentation for this class was generated from the following file:

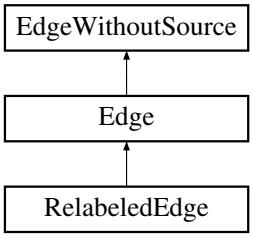
- duplicatesRemover.h

A.3.5 Edge Class Reference

Represents a directed and weighted edge.

`#include <edge.h>`

Inheritance diagram for Edge::



Public Types

- typedef EdgeWeight **key_type**
  Used by stxxl::ksort in order to sort by weight.

Public Methods

- **Edge** (NodeID source=0, NodeID target=0, EdgeWeight weight=0)
  The default constructor.
- NodeID **source** () const
  Returns the identifier of the source vertex.
- void **swap** ()
  Swaps the source and the target vertex.
- bool **isSelfLoop** () const
  Returns true iff this Edge is a self loop, i.e.

Static Public Attributes

- Edge **minWeight** ()
  Returns an Edge object with minimum weight.
- Edge **maxWeight** ()
  Returns an Edge object with maximum weight.

Private Attributes

- NodeID **_source**

Friends

- bool **operator==** (const Edge &e1, const Edge &e2)
  Determines if two edges have the same source and the same target vertex.
- std::ostream & **operator<<** (std::ostream &os, const Edge &e)
  Writes a string representation of an Edge to an output stream.

A.3.5.1 Detailed Description

Represents a directed and weighted edge.

A.3.5.2 Member Function Documentation

bool Edge::isSelfLoop () const `[inline]`

Returns true iff this Edge is a self loop, i.e.

the source and the target vertices are identical.

The documentation for this class was generated from the following file:

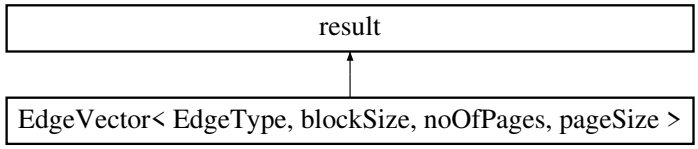
- edge.h

A.3.6 EdgeVector< EdgeType, blockSize, noOfPages, pageSize > Class Template Reference

Represents a weighted graph by a list of Edges.

`#include <edgeVector.h>`

Inheritance diagram for EdgeVector< EdgeType, blockSize, noOfPages, pageSize >::



Public Methods

- **EdgeVector** (NodeCount noOfNodes, EdgeCount noOfEdges)
  Constructs an EdgeVector.
- EdgeCount **noOfEdges** () const
  Returns the number of edges of the graph.
- NodeCount **noOfNodes** () const
  Returns the number of nodes of the graph.
- bool **empty** () const
  Returns true iff this list is empty, i.e.
- void **sortByWeight** ()
  Sorts the edges of the graph by weight.

Private Attributes

- NodeCount **_noOfNodes**

Friends

- std::ostream & **operator**<< (std::ostream &os, EdgeVector< EdgeType, blockSize, noOfPages, pageSize > &el)
  Writes a string representation of the edge list to an output stream.

A.3.6.1 Detailed Description

template<typename EdgeType = Edge, unsigned int blockSize = DS_DEFAULT_BLOCK_SIZE, unsigned int noOfPages = DS_DEFAULT_NO_OF_PAGES, unsigned int pageSize = DS_DEFAULT_PAGE_SIZE> class EdgeVector< EdgeType, blockSize, noOfPages, pageSize >

Represents a weighted graph by a list of Edges.

A.3.6.2 Constructor & Destructor Documentation

template<typename EdgeType = Edge, unsigned int blockSize = DS_DEFAULT_BLOCK_SIZE, unsigned int noOfPages = DS_DEFAULT_NO_OF_PAGES, unsigned int pageSize = DS_DEFAULT_PAGE_SIZE> EdgeVector< EdgeType, blockSize, noOfPages, pageSize >::EdgeVector (NodeCount *noOfNodes*, EdgeCount *noOfEdges*) `[inline]`

Constructs an EdgeVector.

The number of nodes is stored and space for the edges is reserved, but the edges aren't created. In order to add the edges of the graph, use methods like "push_back".

Parameters:

noOfNodes the number of nodes in the graph

noOfEdges the number of edges which space is reserved for

A.3.6.3 Member Function Documentation

template<typename EdgeType = Edge, unsigned int blockSize = DS_DEFAULT_BLOCK_SIZE, unsigned int noOfPages = DS_DEFAULT_NO_OF_PAGES, unsigned int pageSize = DS_DEFAULT_PAGE_SIZE> bool EdgeVector< EdgeType, blockSize, noOfPages, pageSize >::empty () const `[inline]`

Returns true iff this list is empty, i.e.

it contains no edges.

The documentation for this class was generated from the following files:

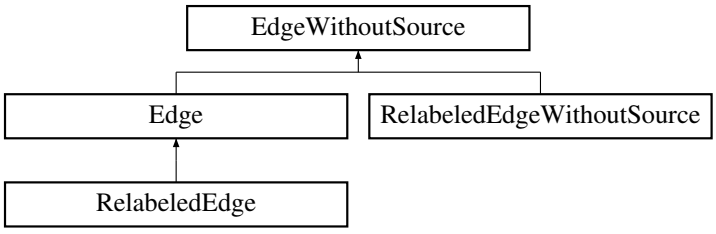
- edgeVector.h
- edgeVector.cpp

A.3.7 EdgeWithoutSource Class Reference

Represents an edge without the source vertex.

`#include <edge.h>`

Inheritance diagram for EdgeWithoutSource::



Public Methods

- **EdgeWithoutSource** (NodeID target=0, EdgeWeight weight=0)

  The default constructor.

- NodeID **target** () const

  Returns the identifier of the target vertex.

- EdgeWeight **weight** () const

  Returns the weight.

Static Public Attributes

- EdgeWithoutSource **minWeight** ()

  Returns an EdgeWithoutSource object with minimum weight.

- EdgeWithoutSource **maxWeight** ()

  Returns an EdgeWithoutSource object with maximum weight.

Protected Methods

- void **setTarget** (NodeID target)

  Sets the target vertex.

Private Attributes

- NodeID **_target**
- EdgeWeight **_weight**

Friends

- bool **operator<** (const EdgeWithoutSource &e1, const EdgeWithoutSource &e2)

  Compares two edges by weight.

- bool **operator==** (const EdgeWithoutSource &e1, const EdgeWithoutSource &e2)

  Returns true iff two EdgesWithoutSource are identical, i.e.

- std::ostream & **operator<<** (std::ostream &os, const EdgeWithoutSource &e)

  Writes a string representation of an EdgeWithoutSource to an output stream.

A.3.7.1 Detailed Description

Represents an edge without the source vertex.

This makes sense when the source vertex is clear from the context.

A.3.7.2 Friends And Related Function Documentation

bool operator== (const EdgeWithoutSource & *e1*, const EdgeWithoutSource & *e2*) `[friend]`

Returns true iff two EdgesWithoutSource are identical, i.e.

the targets and the weights are equal.

The documentation for this class was generated from the following file:

- edge.h

A.3.8 GetWeight< EdgeType > Class Template Reference

Extracts the weight of an edge.

`#include <edge.h>`

Public Types

- typedef Edge::key_type **key_type**

Public Methods

- key_type **operator()** (const EdgeType &obj)

Static Public Methods

- EdgeType **min_value** ()
- EdgeType **max_value** ()

A.3.8.1 Detailed Description

template<typename EdgeType = Edge> class GetWeight< EdgeType >

Extracts the weight of an edge.

Used by stxxl::ksort in order to sort by weight.

The documentation for this class was generated from the following file:

- edge.h

A.3.9 Kruskal< EdgeType > Class Template Reference

Represents Kruskal's algorithm for determining a Minimum Spanning Tree of a weighted graph and the required union/find data structure.

`#include <kruskal.h>`

Public Methods

- **Kruskal** (**EdgeVector**< EdgeType > &graph, **MST** &result)

  Computes a **MST** (p. 42) *of a graph.*

- ~**Kruskal** ()

  The destructor.

Private Methods

- void **computeMST** ()

  Computes a **MST** (p. 42) *of the graph.*

- void **initUnionFind** ()

  Initializes the union/find data structure.

- NodeID **find** (NodeID node)

  Performs a find operation.

- bool **unite** (NodeID node1, NodeID node2)

  Performs a union operation.

Private Attributes

- **EdgeVector**< EdgeType > & **_graph**

  Reference to the given graph.

- **MST** & **_result**

  Reference to a **MST** (p. 42) *object which stores the result.*

- std::vector< NodeID > **_parent**

  A vector which contains for each node the identifier of the parent node.

- std::vector< char > **_height**

  A vector which contains for each node the height of the belonging tree.

- EdgeCount **edgesAddedToResult**

  Counts the number of edges which have been added to the resulting **MST** (p. 42).

A.3.9.1 Detailed Description

template<typename EdgeType = Edge> class Kruskal< EdgeType >

Represents Kruskal's algorithm for determining a Minimum Spanning Tree of a weighted graph and the required union/find data structure.

A.3.9.2 Constructor & Destructor Documentation

template<typename EdgeType = Edge> Kruskal< EdgeType >::Kruskal (EdgeVector< EdgeType > & *graph*, MST & *result*) `[inline]`

Computes a **MST** (p. 42) of a graph.

Parameters:

graph a reference to an **EdgeVector** (p. 36) which represents the graph

result a reference to a **MST** (p. 42) object which stores the result

template<typename EdgeType = Edge> Kruskal< EdgeType >::~Kruskal () `[inline]`

The destructor.

The given graph is deleted.

A.3.9.3 Member Function Documentation

template<typename EdgeType> NodeID Kruskal< EdgeType >::find (NodeID *node*) `[inline, private]`

Performs a find operation.

Path compression is applied.

Parameters:

node the identifier of the node whose set should be determined

Returns :

the identifier of the canonical node which represents the set which "node" belongs to

template<typename EdgeType> bool Kruskal< EdgeType >::unite (NodeID *node1*, NodeID *node2*) `[inline, private]`

Performs a union operation.

Returns :

true iff node1 and node2 have belonged to different sets

A.3.9.4 Member Data Documentation

template<typename EdgeType = Edge> std::vector<char> Kruskal< EdgeType >::_height `[private]`

A vector which contains for each node the height of the belonging tree.

Only the values of canonical nodes are relevant.

template<typename EdgeType = Edge> std::vector<NodeID> Kruskal< EdgeType >::_parent `[private]`

A vector which contains for each node the identifier of the parent node.

The canonical node of a set (= the root of a tree) points to itself.

The documentation for this class was generated from the following files:

- kruskal.h
- kruskal.cpp

A.3.10 MST Class Reference

Represents a Minimum Spanning Tree (MST) of a graph.

`#include <mst.h>`

Public Methods

- **MST** ()

  The default constructor.

- EdgeCount **noOfEdges** () const

  Returns the number of edges of the MST.

- EdgeWeightBig **totalWeight** () const

  Returns the sum of the weights of all edges of the MST.

- void **add** (const **RelabeledEdge** &edge)

  Adds an edge to the MST.

- void **add** (const **RelabeledEdgeWithoutSource** &edge)

  Adds an edge to the MST.

- void **add** (const **Edge** &edge)

  Adds an edge to the MST.

Private Attributes

- **EdgeVector**< **Edge**, DS_MST_BLOCK_SIZE, DS_MST_NO_OF_PAGES, DS_MST_PAGE_SIZE > **_mst**
- EdgeWeightBig **_totalWeight**

Friends

- std::ostream & **operator**<< (std::ostream &os, MST &mst)

  Writes a string representation of the MST to an output stream.

A.3.10.1 Detailed Description

Represents a Minimum Spanning Tree (MST) of a graph.

A.3.10.2 Member Function Documentation

void MST::add (const RelabeledEdgeWithoutSource & *edge*) `[inline]`

Adds an edge to the MST.

This is an explicit copy of the add method for a **RelabeledEdge** (p. 48) in order to avoid virtual methods.

The documentation for this class was generated from the following file:

- mst.h

A.3.11 PQueue Class Reference

Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures.

`#include <pQueue.h>`

Public Methods

- **PQueue** (**EdgeVector**< **Edge** > &graph, **MST** &result, NodeCount noOfNodesInIntMem)

  Reduces the number of nodes of a graph (in order to compute a minimum spanning tree).

- InternalMemoryBucket * **getIntMemBucket** ()

  Returns a pointer to the first external bucket which contains the edges of the nodes which fit in internal memory.

- void **add** (const **RelabeledEdge** &edge)

  Adds the given edge to the first external bucket or to the priority queue depending on the source vertex ID.

Private Types

- typedef stxxl::PRIORITY_QUEUE_GENERATOR< **RelabeledEdge**, **SourceWeightOrdering**< **RelabeledEdge** >, DS_PQUEUE_INTERNAL_MEMORY, DS_PQUEUE_MAX_SIZE >::result **PriorityQueue**

  The type of the priority queue that is used during the node reduction phase.

Private Methods

- void **initPQueue** ()

  Initializes the priority queue and distributes the edges to the first external bucket and the priority queue.

- void **reduceNodes** ()

  Reduces the number of nodes.

Private Attributes

- **EdgeVector< Edge > & _graph**

  Reference to the given graph.

- **MST & _result**

  Reference to a **MST** (p. 42) *object which stores the result.*

- NodeCount **_noOfNodesInIntMem**

  The number of nodes which fit in internal memory.

- InternalMemoryBucket * **_firstExtBucket**

  The first external bucket which contains the edges of the nodes which fit in internal memory.

- stxxl::prefetch_pool< PriorityQueue::block_type > **_prefetchPool**

  The prefetch pool that is used by the priority queue.

- stxxl::write_pool< PriorityQueue::block_type > **_writePool**

  The write pool that is used by the priority queue.

- **PriorityQueue _pqueue**

  The priority queue.

A.3.11.1 Detailed Description

Represents an algorithm for reducing the number of nodes of a graph (in order to compute a minimum spanning tree) and the required data structures.

This implementation uses an external priority queue.

A.3.11.2 Constructor & Destructor Documentation

PQueue::PQueue (EdgeVector< Edge > & *graph*, MST & *result*, NodeCount *noOfNodesInIntMem*) `[inline]`

Reduces the number of nodes of a graph (in order to compute a minimum spanning tree).

Parameters:

graph a reference to an **EdgeVector** (p. 36) which represents the graph

result a reference to a **MST** (p. 42) object which stores the result

noOfNodesInIntMem the number of nodes which fit in internal memory

A.3.11.3 Member Function Documentation

void PQueue::reduceNodes () `[private]`

Reduces the number of nodes.

Only the first external bucket ”survives”.

The documentation for this class was generated from the following files:

- pQueue.h
- pQueue.cpp

A.3.12 Randomizer< Bijection > Class Template Reference

A bijection over {0,...,n-1} that can be used to randomize an **Edge** (p. 34) in order to obtain a (pseudo-)random permutation.

`#include <randomizer.h>`

Public Methods

- **Randomizer** (NodeCount noOfNodes)

  The constructor.

- **RelabeledEdge randomize** (const **Edge** &edge) const

  Randomizes an **Edge** (p. 34)*.*

Private Methods

- NodeID **randomize** (NodeID nodeID) const

  Randomizes a node ID using the underlying bijection.

Private Attributes

- NodeCount **_noOfNodes**

  The number of nodes that specifies the domain and co-domain of the bijection.

- Bijection **_bijection**

  The underlying bijection.

A.3.12.1 Detailed Description

template<typename Bijection = RandomizerLinearCongruence> class Randomizer< Bijection >

A bijection over {0,...,n-1} that can be used to randomize an **Edge** (p. 34) in order to obtain a (pseudo-)random permutation.

Either **RandomizerLinearCongruence** (p. 47) or **RandomizerFeistel** (p. 46) can be used as underlying bijection.

A.3.12.2 Member Function Documentation

template<typename Bijection = RandomizerLinearCongruence> RelabeledEdge Randomizer< Bijection >::randomize (const Edge & *edge*) const `[inline]`

Randomizes an **Edge** (p. 34).

The source and the target vertices are randomized using the bijection. The original source and target vertices are saved, so a **RelabeledEdge** (p. 48), which contains both the randomized and the original vertices, is returned.

The documentation for this class was generated from the following file:

- randomizer.h

A.3.13 RandomizerFeistel Class Reference

A bijection that uses Feistel permutations.

`#include <randomizer.h>`

Public Methods

- **RandomizerFeistel** (NodeCount noOfNodes)
  The constructor.
- NodeID **operator()** (NodeID nodeID) const
  The bijection.

Private Methods

- void **initRandomNumbers** ()
  Initializes the table of random numbers.

Private Attributes

- NodeID **_sqRoot**
  The next integer >= the square root of the given number of nodes.
- int **randomNumbers** [**_noOfIterations**][**_maxSqRoot**]
  The table of random numbers (used by the Feistel permutations).

Static Private Attributes

- const int **_noOfIterations** = 2
  The number of performed Feistel permutations.
- const int **_maxSqRoot** = 0x10000
  The maximum size of _sqRoot.

A.3.13.1 Detailed Description

A bijection that uses Feistel permutations.

A.3.13.2 Constructor & Destructor Documentation

RandomizerFeistel::RandomizerFeistel (NodeCount *noOfNodes*) `[inline]`

The constructor.

A bijection over {0,...,r^2-1} is initialized, whereat r is the next integer greater than or equal to the square root of a given n.

The documentation for this class was generated from the following file:

- randomizer.h

A.3.14 RandomizerLinearCongruence Class Reference

A bijection that uses the linear congruential method.

`#include <randomizer.h>`

Public Methods

- **RandomizerLinearCongruence** (NodeCount noOfNodes)

  The constructor.

- NodeID **operator()** (NodeID nodeID) const

  The bijection.

Private Methods

- void **determineNextPrime** (NodeCount noOfNodes)

  Determines the next prime number >= the given number of nodes.

- bool **isPrime** (NodeCount p) const

  Returns true iff the given number is prime.

Private Attributes

- NodeCount **_prime**

  The next prime number >= the given number of nodes.

A.3.14.1 Detailed Description

A bijection that uses the linear congruential method.

A.3.14.2 Constructor & Destructor Documentation

RandomizerLinearCongruence::RandomizerLinearCongruence (NodeCount *noOfNodes*) `[inline]`

The constructor.

A bijection over {0,...p-1} is initialized, whereat p is the next prime number greater than or equal to a given n.

The documentation for this class was generated from the following file:

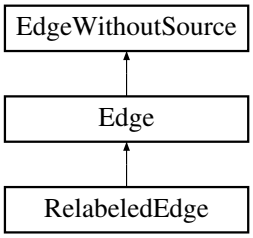
- randomizer.h

A.3.15 RelabeledEdge Class Reference

Represents a relabeled edge.

`#include <relabeledEdge.h>`

Inheritance diagram for RelabeledEdge::



Public Methods

- **RelabeledEdge** (NodeID source=0, NodeID target=0, EdgeWeight weight=0, NodeID origSource=0, NodeID origTarget=0)

  The default constructor.

- **RelabeledEdge** (const **Edge** &edge)

  Creates a relabeled edge from a normal edge.

- **RelabeledEdge** (const **RelabeledEdgeWithoutSource** &edge, NodeID newSource)

  Creates a relabeled edge from a **RelabeledEdgeWithoutSource** (p. 49).

- NodeID **originalSource** () const

  Returns the identifier of the original source vertex.

- NodeID **originalTarget** () const

  Returns the identifier of the original target vertex.

Static Public Attributes

- RelabeledEdge **minWeight** ()

Returns a RelabeledEdge object with minimum weight.

- RelabeledEdge **maxWeight** ()

  Returns a RelabeledEdge object with maximum weight.

Private Attributes

- NodeID **_originalSource**
- NodeID **_originalTarget**

Friends

- std::ostream & **operator<<** (std::ostream &os, const RelabeledEdge &e)

  Writes a string representation of a RelabeledEdge to an output stream.

A.3.15.1 Detailed Description

Represents a relabeled edge.

I.e. an edge whose source and target vertices have been relabeled and the original indices are stored additionally. This class doesn't extend **RelabeledEdgeWithoutSource** (p. 49) in order to avoid multiple inheritance.

A.3.15.2 Constructor & Destructor Documentation

RelabeledEdge::RelabeledEdge (const Edge & *edge*) `[inline]`

Creates a relabeled edge from a normal edge.

The original source resp. target equals the current source resp. target.

RelabeledEdge::RelabeledEdge (const RelabeledEdgeWithoutSource & *edge*, NodeID *newSource*) `[inline]`

Creates a relabeled edge from a **RelabeledEdgeWithoutSource** (p. 49).

The source vertex must be provided as second parameter. If necessary, source and target are swapped, so that source is greater than (or equal to) target.

The documentation for this class was generated from the following file:

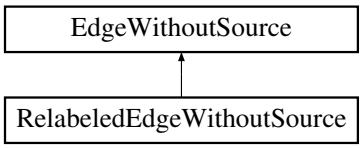
- relabeledEdge.h

A.3.16 RelabeledEdgeWithoutSource Class Reference

Represents a relabeled edge without the source vertex.

`#include <relabeledEdge.h>`

Inheritance diagram for RelabeledEdgeWithoutSource::



Public Methods

- **RelabeledEdgeWithoutSource** ()

  The default constructor.

- **RelabeledEdgeWithoutSource** (const **RelabeledEdge** &edge)

  Creates a RelabeledEdgeWithoutSource from a **RelabeledEdge** (p. 48).

- NodeID **originalSource** () const

  Returns the identifier of the original source vertex.

- NodeID **originalTarget** () const

  Returns the identifier of the original target vertex.

Private Attributes

- NodeID **_originalSource**
- NodeID **_originalTarget**

Friends

- std::ostream & **operator**<< (std::ostream &os, const RelabeledEdgeWithoutSource &e)

  Writes a string representation of a RelabeledEdgeWithoutSource to an output stream.

A.3.16.1 Detailed Description

Represents a relabeled edge without the source vertex.

This makes sense when the source vertex is clear from the context.

A.3.16.2 Constructor & Destructor Documentation

RelabeledEdgeWithoutSource::RelabeledEdgeWithoutSource (const RelabeledEdge & *edge*)

Creates a RelabeledEdgeWithoutSource from a **RelabeledEdge** (p. 48).

The source vertex is thrown away.

The documentation for this class was generated from the following file:

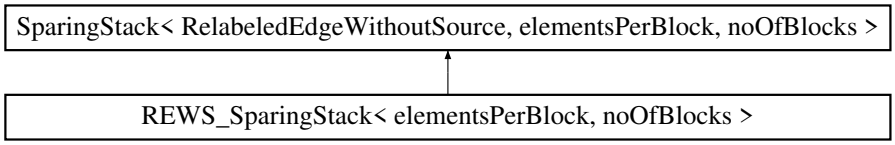
- relabeledEdge.h

A.3.17 REWS_SparingStack< elementsPerBlock, noOfBlocks > Class Template Reference

A specialized **SparingStack** (p. 53) which stores **RelabeledEdgeWithoutSource** (p. 49)-objects.

`#include <sparingStack.h>`

Inheritance diagram for REWS_SparingStack< elementsPerBlock, noOfBlocks >::



Public Methods

- NodeID **determineMinEdge** (**MST** &result) const

  Determines the edge with minimum weight.

A.3.17.1 Detailed Description

template<int elementsPerBlock, int noOfBlocks> class REWS_SparingStack< elementsPerBlock, noOfBlocks >

A specialized **SparingStack** (p. 53) which stores **RelabeledEdgeWithoutSource** (p. 49)-objects.

A.3.17.2 Member Function Documentation

template<int elementsPerBlock, int noOfBlocks> NodeID REWS_SparingStack< elementsPerBlock, noOfBlocks >::determineMinEdge (MST & *result*) const `[inline]`

Determines the edge with minimum weight.

It is added to the resulting minimum spanning tree and the target node ID is returned.

Parameters:

result a reference to the **MST** (p. 42) object which stores the resulting minimum spanning tree

The documentation for this class was generated from the following file:

- sparingStack.h

A.3.18 SourceTargetWeightOrdering Class Reference

A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by the target vertex and finally by weight.

`#include <edge.h>`

Public Methods

- bool **operator()** (const **Edge** &e1, const **Edge** &e2) const

Static Public Attributes

- **Edge min_value** ()
- **Edge max_value** ()

A.3.18.1 Detailed Description

A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by the target vertex and finally by weight.

The documentation for this class was generated from the following file:

- edge.h

A.3.19 SourceWeightOrdering< EdgeType > Class Template Reference

A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by weight.

`#include <edge.h>`

Public Methods

- bool **operator()** (const EdgeType &e1, const EdgeType &e2) const

Static Public Methods

- EdgeType **min_value** ()
- EdgeType **max_value** ()

A.3.19.1 Detailed Description

template<typename EdgeType = Edge> class SourceWeightOrdering< EdgeType >

A StrictWeakOrdering predicate which can be used to compare two edges first by the source vertex, then - if the source vertices are equal - by weight.

Used as comparison type by stxxl::priority_queue. The largest element, which is returned by top(), is the edge with the highest source vertex ID and (if there are several edges with the same source vertex) minimum weight.

The documentation for this class was generated from the following file:

- edge.h

A.3.20 SparingStack< value_type, elementsPerBlock, noOfBlocks > Class Template Reference

A stack which stores its elements in blocks which are managed by a **CommonPoolOfBlocks** (p. 29).

`#include <sparingStack.h>`

Public Methods

- **SparingStack** ()

  The default constructor.

- ∼**SparingStack** ()

  The destructor.

- void **setPool** (**Pool** ∗pool)

  Sets the **CommonPoolOfBlocks** (p. 29) *which should be used.*

- bool **empty** () const

  Returns true iff the stack is empty.

- void **push** (const value_type &element)

  Adds an element to the stack.

- const value_type & **top** () const

  Returns a reference to the last element.

- void **pop** ()

  Removes the last element.

- int **size** () const

  Returns the number of elements which are stored in the stack.

Protected Types

- typedef **CommonPoolOfBlocks**< value_type, elementsPerBlock, noOfBlocks > **Pool**

  The type of the used **CommonPoolOfBlocks** (p. 29).

- typedef Pool::Block **Block**

  The type of a Block which stores several elements.

Protected Attributes

- **Pool** ∗ **_pool**

  A pointer to the used **CommonPoolOfBlocks** (p. 29).

- **Block** ∗ **_top**

  A pointer to the list of blocks which store the elements.

- **Block _bottom**

  The bottom block which belongs to the stack.

A.3.20.1 Detailed Description

template<typename value_type, int elementsPerBlock, int noOfBlocks> class SparingStack< value_type, elementsPerBlock, noOfBlocks >

A stack which stores its elements in blocks which are managed by a **CommonPoolOfBlocks** (p. 29).

A.3.20.2 Member Function Documentation

template<typename value_type, int elementsPerBlock, int noOfBlocks> void SparingStack< value_type, elementsPerBlock, noOfBlocks >::setPool (Pool * *pool*) `[inline]`

Sets the **CommonPoolOfBlocks** (p. 29) which should be used.

This method must be called after the object has been created and before it is used.

The documentation for this class was generated from the following file:

- sparingStack.h

A.3.21 WeightOrdering< EdgeType > Class Template Reference

A StrictWeakOrdering predicate which can be used to compare two edges by weight.

`#include <edge.h>`

Public Methods

- bool **operator()** (const EdgeType &e1, const EdgeType &e2) const

Static Public Methods

- EdgeType **min_value** ()
- EdgeType **max_value** ()

A.3.21.1 Detailed Description

template<typename EdgeType = Edge> class WeightOrdering< EdgeType >

A StrictWeakOrdering predicate which can be used to compare two edges by weight.

Used by stxxl::sort in order to sort by weight.

The documentation for this class was generated from the following file:

- edge.h

Appendix B

Source Code

B.1 Main

B.1.1 config.h

```
// ** COMMON SETTINGS **
// display verbose debugging messages
#define DS_VERBOSE1(x) x
#define DS_VERBOSE2(x)

// logging
#define DS_LOGGING1(x) x
#define DS_LOGGING2(x) x

// external sorting
#define DS_INTERNAL_MEMORY_FOR_SORTING 650*1024*1024

// default template parameters of EdgeVector
// used by the first external bucket and by the input graph
#define DS_DEFAULT_BLOCK_SIZE 2*1024*1024
#define DS_DEFAULT_NO_OF_PAGES 1
#define DS_DEFAULT_PAGE_SIZE 4
#define DS_DEFAULT_PAGER lru_pager

// template parameters of the mst object which stores the result
#define DS_MST_BLOCK_SIZE 2*1024*1024
#define DS_MST_NO_OF_PAGES 1
#define DS_MST_PAGE_SIZE 4

// remove duplicates
#define DS_REMOVE_DUPLICATES(x) x
#define DS_DONT_REMOVE_DUPLICATES(x)

// randomize the input graph
#define DS_RANDOMIZE(x) x
#define DS_DONT_RANDOMIZE(x)

// node reduction implementation
#define DS_USE_BUCKETS(x) x
#define DS_USE_PQUEUE(x)

// ** BUCKETS IMPLEMENTATION **
// external buckets
#define DS_EXT_BUCKET_SIZE 1800000
// (The size of the first external bucket is the number of nodes which
//  fit in internal memory.)
```

```
#define DS_EXT_FIRST_BUCKET_SIZE 160*1000*1000

// template parameters of the external buckets
// (these values don't apply to the first external bucket)
#define DS_EXT_BLOCK_SIZE 512*1024
#define DS_EXT_PAGE_SIZE 1
#define DS_EXT_PREFETCH_POOL 4
#define DS_EXT_WRITE_POOL 4

// template parameters of the internal buckets
#define DS_INT_EDGES_PER_BLOCK 8
#define DS_INT_NO_OF_BLOCKS 1500000


// ** PQUEUE IMPLEMENTATION **
#define DS_PQUEUE_INTERNAL_MEMORY 500*1024*1024
#define DS_PQUEUE_MAX_SIZE 3*1000*1000
#define DS_PQUEUE_PREFETCH_POOL 50
#define DS_PQUEUE_WRITE_POOL 50
```

B.1.2 main.cpp

```
#include <iostream>


#include "config.h"
#include "data_structures/mst.h"
#include "data_structures/edgeVector.cpp"
#include "utils/logging.h"
#include "utils/utils.h"

DS_LOGGING1( Logging logging("log.txt") );

#include "buckets/buckets.cpp"
#include "priority_queue/pQueue.cpp"
#include "utils/generate.cpp"
#include "utils/importExport.cpp"
#include "base_case/kruskal.cpp"
#include "utils/parameters.h"


/** Aborts processing after the node reduction phase. */
void abortAfterNodeReducing(std::string filename, InternalMemoryBucket *reducedGraph)
{
    if (filename != "") {
        // write reduced graph to the given file
        std::ofstream outFile(filename.c_str());
        // dummy EdgeVector in order to avoid a compiler error:
        // An instance of EdgeVector<RelabeledEdge> is needed to provide
        // a suitable output operator for intMemBucket.
        EdgeVector<RelabeledEdge> dummy(0,0);
        outFile << 'r' << std::endl << *reducedGraph;
        DS_LOGGING1( logging.events().push_back(
                        LoggingEvent("reduced graph written to file")) );
    }
    std::cout << std::endl << "abort after the 'node reducing'-phase" << std::endl;
}


/** The main program. */
int main(int argc, char *argv[])
{
    DS_LOGGING1( logging.events().push_back(LoggingEvent("begin")) );
```

```
Parameters param(argc, argv); // parse the command-line arguments

DS_USE_BUCKETS(
    typedef Buckets<DS_EXT_BUCKET_SIZE> MyBuckets;
    // compute the number of buckets that are needed to store the edges of all nodes
    MyBuckets::BucketID noOfBuckets =
        MyBuckets::noOfExtBuckets(param.noOfNodes(),
                                  param.noOfNodesInInternalMemory());

    DS_VERBOSE1( std::cout << "number of external buckets = "
                           << noOfBuckets << std::endl;
                 memoryUsageForecast(param.noOfNodes(),
                                     param.noOfNodesInInternalMemory(),
                                     noOfBuckets) );
)

// INPUT
EdgeVector<> * inputGraph;

if (param.randomGraph()) {
    // generate random graph
    inputGraph = generateRandomGraph(param.randomGraphNoOfNodes(),
                                     param.randomGraphNoOfEdges());

    DS_LOGGING1( logging.setProblemInstance(
                    new LoggingRandomGraph(param.randomGraphNoOfNodes(),
                                           param.randomGraphNoOfEdges(),
                                           param.noOfNodesInInternalMemory())) );
}
if (param.gridGraph()) {
    // generate grid graph
    inputGraph = generateGridGraph(param.gridGraphNoOfNodesX(),
                                   param.gridGraphNoOfNodesY());

    DS_LOGGING1( logging.setProblemInstance(
                    new LoggingGridGraph(param.gridGraphNoOfNodesX(),
                                         param.gridGraphNoOfNodesY(),
                                         param.noOfNodesInInternalMemory())) );
}
if (param.geometricGraph()) {
    // generate geometric graph
    inputGraph = generateGeometricGraph(param.geometricGraphNoOfNodes(),
                                        param.geometricGraphNoOfNeighbours());

    DS_LOGGING1( logging.setProblemInstance(
                    new LoggingGeometricGraph(param.geometricGraphNoOfNodes(),
                                              param.geometricGraphNoOfNeighbours(),
                                              inputGraph->noOfEdges(),
                                              param.noOfNodesInInternalMemory())) );
}
if (param.importInputFilename() != "") {
    // import graph from file
    DS_VERBOSE1( std::cout << std::endl << "import graph" << std::endl );
    inputGraph = importEdgeVector( param.importInputFilename() );

    DS_LOGGING1( logging.setProblemInstance(
                    new LoggingImportedGraph(inputGraph->noOfNodes(),
                                             inputGraph->noOfEdges(),
                                             param.noOfNodesInInternalMemory(),
                                             param.importInputFilename())) );
}

// export input graph
if (param.exportInputFilename() != "") {
    DS_VERBOSE1( std::cout << std::endl << "export graph" << std::endl );

    if ( param.exportInputFormatAdjacencyList() ) {
```

```
        // export format: adjacency lists
        std::ofstream outFile(param.exportInputFilename().c_str());
        exportEdgeVectorAdjList(outFile, *inputGraph);
    }
    else {
        if ( param.exportInputFormatCompressed() ) {
            // export format: compressed
            exportEdgeVectorCompressed(param.exportInputFilename(), *inputGraph);
        }
        else {
            // export format: list of edges
            std::ofstream outFile(param.exportInputFilename().c_str());
            outFile << *inputGraph;
        }
    }
    if ( param.quitAfterExporting() ) {
        // quit after exporting
        std::cout << std::endl
                  << "quit after exporting the input graph." << std::endl;
        exit(0);
    }
}

DS_LOGGING1( logging.events().push_back(LoggingEvent("graph generated/imported")) );



MST result; // the resulting MST

// PROCESSING
if ( param.noOfNodes() <= param.noOfNodesInInternalMemory() ) {
    // all nodes fit in internal memory
    // ==> Kruskal's algorithm can by applied immediately
    Kruskal<Edge> *kruskal = new Kruskal<Edge>(*inputGraph,result);
    delete kruskal;
}
else {
    // node reduction phase is required
    DS_LOGGING1( logMemoryUsage("before buckets") );


    // perform node reduction

    DS_USE_BUCKETS(
        // buckets implementation
        MyBuckets *buckets = new MyBuckets(*inputGraph, result,
                                           noOfBuckets,
                                           param.noOfNodesInInternalMemory());
        InternalMemoryBucket *intMemBucket = buckets->getIntMemBucket();
        delete buckets;
    )

    DS_USE_PQUEUE(
        // priority queue implementation
        PQueue *pqueue = new PQueue(*inputGraph, result,
                                    param.noOfNodesInInternalMemory());
        InternalMemoryBucket *intMemBucket = pqueue->getIntMemBucket();
        delete pqueue;
    )

    // end of 'node reduction'


    DS_LOGGING1( logMemoryUsage("after buckets") );

    DS_LOGGING1( logging.events().push_back(LoggingEvent("nodes reduced")) );
```

```
        if ( param.abortAfterNodeReducing() ) {
            // abort processing after the node reduction phase
            abortAfterNodeReducing( param.reducedGraphFilename(), intMemBucket );
        }
        else {
            // apply Kruskal's algorithm to the reduced graph
            Kruskal<RelabeledEdge> *kruskal =
                new Kruskal<RelabeledEdge>(*intMemBucket,result);
            delete kruskal;
        }
    }



    // OUTPUT
    DS_LOGGING1( logging.setResult(new LoggingResult(result.noOfEdges(),
                                                     result.totalWeight())) );
    DS_LOGGING1( logging.events().push_back(LoggingEvent("end")) );
    DS_LOGGING1( logging.printLog() );

    if (param.outputFilename() != "") {
        // write the computed mst to the given file
        std::ofstream outFile(param.outputFilename().c_str());
        outFile << result;
    }

    DS_VERBOSE1( std::cout << std::endl << "finished." << std::endl );

    return 0;
}
```

B.2 Data Structures

B.2.1 edge.h

```
#ifndef EDGE_H
#define EDGE_H

#include <iostream>

typedef unsigned int NodeID;
typedef unsigned int EdgeWeight;
typedef unsigned long long int EdgeWeightBig; // used for the sum of edge weights

/**
   Represents an edge without the source vertex.
   This makes sense when the source vertex is clear from the context.
*/
class EdgeWithoutSource
{
    /** Compares two edges by weight. */
    friend bool operator<(const EdgeWithoutSource &e1,
                          const EdgeWithoutSource &e2) {
        return e1._weight<e2._weight;
    }

    /**
       Returns true iff two EdgesWithoutSource are identical,
       i.e. the targets and the weights are equal.
    */
    friend bool operator==(const EdgeWithoutSource &e1,
```

```
                            const EdgeWithoutSource &e2) {
        return (e1._target == e2._target) && (e1._weight == e2._weight);
    }

    /** Writes a string representation of an EdgeWithoutSource to an output stream. */
    friend std::ostream& operator<<( std::ostream& os, const EdgeWithoutSource &e ) {
        os << e.target() << " " << e.weight();
        return os;
    }

 public:
    /** Returns an EdgeWithoutSource object with minimum weight. */
    static EdgeWithoutSource minWeight() {return EdgeWithoutSource(0,0);}

    /** Returns an EdgeWithoutSource object with maximum weight. */
    static EdgeWithoutSource maxWeight() {return EdgeWithoutSource(0,0xffffffff);}

    /** The default constructor. */
    EdgeWithoutSource( NodeID target = 0, EdgeWeight weight = 0 )
        : _target( target ), _weight( weight )
        {}

    /** Returns the identifier of the target vertex. */
    NodeID target() const {return _target;}

    /** Returns the weight. */
    EdgeWeight weight() const {return _weight;}

 protected:
    /** Sets the target vertex. */
    void setTarget(NodeID target) {_target = target;}

 private:
    NodeID _target;
    EdgeWeight _weight;
};

/**
   Represents a directed and weighted edge.
*/
class Edge : public EdgeWithoutSource
{
    /** Determines if two edges have the same source and the same target vertex. */
    friend bool operator==(const Edge &e1, const Edge &e2) {
        return ( e1.source()==e2.source() ) && ( e1.target()==e2.target() );
    }

    /** Writes a string representation of an Edge to an output stream. */
    friend std::ostream& operator<<( std::ostream& os, const Edge &e ) {
        os << e.source() << " " << e.target() << " " << e.weight();
        return os;
    }

 public:
    /** Used by stxxl::ksort in order to sort by weight. */
    typedef EdgeWeight key_type;


    /** Returns an Edge object with minimum weight. */
    static Edge minWeight() {return Edge(0,0,0);}

    /** Returns an Edge object with maximum weight. */
    static Edge maxWeight() {return Edge(0,0,0xffffffff);}


    /** The default constructor. */
    Edge(NodeID source = 0, NodeID target = 0, EdgeWeight weight = 0)
```

```
        : _source( source ), EdgeWithoutSource( target, weight )
        {}

    /** Returns the identifier of the source vertex. */
    NodeID source() const {return _source;}

    /** Swaps the source and the target vertex. */
    void swap() {NodeID tmp = _source; _source = target(); setTarget(tmp);}

    /**
       Returns true iff this Edge is a self loop, i.e. the source and the
       target vertices are identical.
    */
    bool isSelfLoop() const {return (source() == target());}


 private:
    NodeID _source;
};



/**
   Extracts the weight of an edge.
   Used by stxxl::ksort in order to sort by weight.
*/
template <typename EdgeType = Edge>
class GetWeight
{
public:
    typedef Edge::key_type key_type;
    key_type operator() (const EdgeType & obj)
        {
            return obj.weight();
        }
    static EdgeType min_value(){ return EdgeType::minWeight(); }
    static EdgeType max_value(){ return EdgeType::maxWeight(); }
};



/**
   A StrictWeakOrdering predicate which can be used to compare two edges by
   weight.
   Used by stxxl::sort in order to sort by weight.
*/
template <typename EdgeType = Edge>
class WeightOrdering
{
 public:
    bool operator() (const EdgeType &e1, const EdgeType &e2) const
        {
            if (e1.weight() < e2.weight()) return true;
            return false;
        }
    static EdgeType min_value(){ return EdgeType::minWeight(); }
    static EdgeType max_value(){ return EdgeType::maxWeight(); }
};


/**
   A StrictWeakOrdering predicate which can be used to compare two edges first by
   the source vertex, then - if the source vertices are equal - by the target
   vertex and finally by weight.
*/
class SourceTargetWeightOrdering
{
```

```
 public:
    bool operator() (const Edge &e1, const Edge &e2) const
        {
            // first, compare by source
            if (e1.source() < e2.source()) return true;
            if (e1.source() > e2.source()) return false;

            // then, by target
            if (e1.target() < e2.target()) return true;
            if (e1.target() > e2.target()) return false;

            // finally, by weight
            if (e1.weight() < e2.weight()) return true;
            return false;
        }

    static Edge min_value() {return Edge(0,0,0);}
    static Edge max_value() {return Edge(0xffffffff,0xffffffff,0xffffffff);}
};


/**
   A StrictWeakOrdering predicate which can be used to compare two edges first by
   the source vertex, then - if the source vertices are equal - by weight.
   Used as comparison type by stxxl::priority_queue. The largest element, which is
   returned by top(), is the edge with the highest source vertex ID and (if there
   are several edges with the same source vertex) minimum weight.
*/
template <typename EdgeType = Edge>
class SourceWeightOrdering
{
 public:
    bool operator() (const EdgeType &e1, const EdgeType &e2) const
        {
            // first, compare by source
            if (e1.source() < e2.source()) return true;
            if (e1.source() > e2.source()) return false;

            // then, by weight
            if (e1.weight() > e2.weight()) return true;
            return false;
        }

    static EdgeType min_value(){ return EdgeType::minWeight(); }
    static EdgeType max_value(){ return EdgeType::maxWeight(); }
};

#endif // EDGE_H
```

B.2.2 relabeledEdge.h

```
#ifndef RELABELEDEDGE_H
#define RELABELEDEDGE_H


#include "edge.h"


class RelabeledEdge;

/**
   Represents a relabeled edge without the source vertex.
   This makes sense when the source vertex is clear from the context.
*/
```

```
class RelabeledEdgeWithoutSource : public EdgeWithoutSource
{
    /**
       Writes a string representation of a RelabeledEdgeWithoutSource
       to an output stream.
    */
    friend std::ostream& operator<<( std::ostream& os,
                                     const RelabeledEdgeWithoutSource &e ) {
        os << e.target() << " " << e.weight() << " "
           << e.originalSource() << " " << e.originalTarget();
        return os;
    }

 public:
    /** The default constructor. */
    RelabeledEdgeWithoutSource()
        : EdgeWithoutSource(), _originalSource( 0 ), _originalTarget( 0 )
        {}

    /**
       Creates a RelabeledEdgeWithoutSource from a RelabeledEdge.
       The source vertex is thrown away.
    */
    RelabeledEdgeWithoutSource( const RelabeledEdge &edge );

    /** Returns the identifier of the original source vertex. */
    NodeID originalSource() const {return _originalSource;}

    /** Returns the identifier of the original target vertex. */
    NodeID originalTarget() const {return _originalTarget;}

 private:
    NodeID _originalSource;
    NodeID _originalTarget;
};


/**
   Represents a relabeled edge. I.e. an edge whose source and target vertices
   have been relabeled and the original indices are stored additionally.
   This class doesn't extend RelabeledEdgeWithoutSource in order to avoid
   multiple inheritance.
*/
class RelabeledEdge : public Edge
{
    /** Writes a string representation of a RelabeledEdge to an output stream. */
    friend std::ostream& operator<<( std::ostream& os, const RelabeledEdge &e ) {
        os << e.source() << " " << e.target() << " " << e.weight() << " "
           << e.originalSource() << " " << e.originalTarget();
        return os;
    }

 public:
    /** Returns a RelabeledEdge object with minimum weight. */
    static RelabeledEdge minWeight() {return RelabeledEdge(0,0,0,0,0);}

    /** Returns a RelabeledEdge object with maximum weight. */
    static RelabeledEdge maxWeight() {return RelabeledEdge(0,0,0xffffffff,0,0);}


    /** The default constructor. */
    RelabeledEdge(NodeID source = 0, NodeID target = 0, EdgeWeight weight = 0,
                  NodeID origSource = 0, NodeID origTarget = 0)
        : Edge( source,target,weight ),
          _originalSource( origSource ), _originalTarget( origTarget )
        {}
```

```
    /**
       Creates a relabeled edge from a normal edge.
       The original source resp. target equals the current source resp. target.
    */
    RelabeledEdge(const Edge &edge)
        : Edge( edge ), _originalSource( edge.source() ),
                        _originalTarget( edge.target() )
        {}

    /**
       Creates a relabeled edge from a RelabeledEdgeWithoutSource.
       The source vertex must be provided as second parameter.
       If necessary, source and target are swapped, so that
       source is greater than (or equal to) target.
    */
    RelabeledEdge(const RelabeledEdgeWithoutSource &edge, NodeID newSource)
        : Edge( newSource, edge.target(), edge.weight() ),
          _originalSource( edge.originalSource() ), _originalTarget( edge.originalTarget() )
        {
            if (source() < target()) swap();
        }

    /** Returns the identifier of the original source vertex. */
    NodeID originalSource() const {return _originalSource;}

    /** Returns the identifier of the original target vertex. */
    NodeID originalTarget() const {return _originalTarget;}


 private:
    NodeID _originalSource;
    NodeID _originalTarget;
};


/**
   Creates a RelabeledEdgeWithoutSource from a RelabeledEdge.
   The source vertex is thrown away.
*/
RelabeledEdgeWithoutSource::RelabeledEdgeWithoutSource( const RelabeledEdge &edge )
    : EdgeWithoutSource( edge.target(), edge.weight() ),
      _originalSource( edge.originalSource() ), _originalTarget( edge.originalTarget() )
{}


#endif // RELABELEDEDGE_H
```

B.2.3 edgeVector.h

```
#ifndef EDGEVECTOR_H
#define EDGEVECTOR_H

#include <iostream>

#include "relabeledEdge.h"

#include "../stxxl/containers/vector.h"

typedef unsigned int EdgeCount;
typedef unsigned int NodeCount;


/**
```

```
   Represents a weighted graph by a list of Edges.
*/
template <typename EdgeType = Edge,
          unsigned int blockSize = DS_DEFAULT_BLOCK_SIZE,
          unsigned int noOfPages = DS_DEFAULT_NO_OF_PAGES,
          unsigned int pageSize = DS_DEFAULT_PAGE_SIZE>
class EdgeVector : public stxxl::VECTOR_GENERATOR<EdgeType, pageSize, noOfPages,
                                                  blockSize>::result
{
    /** Writes a string representation of the edge list to an output stream. */
    friend std::ostream& operator<<( std::ostream& os,
                                     EdgeVector<EdgeType,blockSize,noOfPages,pageSize> &el ) {
        os << el.noOfNodes() << " " << el.noOfEdges() << std::endl;
        for (EdgeCount i=0; i<el.noOfEdges(); i++)
            os << el[i] << std::endl;
        return os;
    }

 public:
    /**
       Constructs an EdgeVector.
       The number of nodes is stored and space for the edges is reserved,
       but the edges aren't created. In order to add the edges of the graph,
       use methods like "push_back".
       @param noOfNodes the number of nodes in the graph
       @param noOfEdges the number of edges which space is reserved for
    */
    EdgeVector(NodeCount noOfNodes, EdgeCount noOfEdges)
        : _noOfNodes( noOfNodes )
        { reserve(noOfEdges); }

    /** Returns the number of edges of the graph. */
    EdgeCount noOfEdges() const {return size();}

    /** Returns the number of nodes of the graph. */
    NodeCount noOfNodes() const {return _noOfNodes;}

    /** Returns true iff this list is empty, i.e. it contains no edges. */
    bool empty() const {return (noOfEdges() == 0);}


    /** Sorts the edges of the graph by weight. */
    void sortByWeight();

 private:
    NodeCount _noOfNodes;

};


/**
   The type of the first external bucket which contains the
   edges of the nodes which fit in internal memory.
*/
typedef EdgeVector<RelabeledEdge> InternalMemoryBucket;


#endif // EDGEVECTOR_H
```

B.2.4 edgeVector.cpp

```
#include "edgeVector.h"
```

```
//#include "../stxxl/algo/sort.h"
#include "../stxxl/algo/ksort.h"

/** Sorts the edges of the graph by weight. */
template <typename EdgeType, unsigned int blockSize,
          unsigned int noOfPages, unsigned int pageSize>
void EdgeVector<EdgeType,blockSize,noOfPages,pageSize>::sortByWeight()
{
    // sort of STXXL
    // stxxl::sort(begin(),end(),WeightOrdering<EdgeType>(),DS_INTERNAL_MEMORY_FOR_SORTING);

    // ksort of STXXL
    stxxl::ksort(begin(),end(),GetWeight<EdgeType>(),DS_INTERNAL_MEMORY_FOR_SORTING);
}
```

B.2.5 mst.h

```
#ifndef MST_H
#define MST_H

#include <iostream>

#include "edgeVector.h"
#include "relabeledEdge.h"

/**
   Represents a Minimum Spanning Tree (MST) of a graph.
*/
class MST
{
    /** Writes a string representation of the MST to an output stream. */
    friend std::ostream& operator<<( std::ostream& os, MST &mst ) {
        os << mst.noOfEdges() << std::endl;
        for (EdgeCount i=0; i<mst.noOfEdges(); i++)
            os << mst._mst[i] << std::endl;
        return os;
    }

 public:
    /** The default constructor. */
    MST()
        : _mst( 0, 0 ), _totalWeight( 0 )
        {}

    /** Returns the number of edges of the MST. */
    EdgeCount noOfEdges() const {return _mst.noOfEdges();}

    /** Returns the sum of the weights of all edges of the MST. */
    EdgeWeightBig totalWeight() const {return _totalWeight;}

    /** Adds an edge to the MST. */
    void add(const RelabeledEdge &edge);

    /** Adds an edge to the MST. */
    void add(const RelabeledEdgeWithoutSource &edge);

    /** Adds an edge to the MST. */
    void add(const Edge &edge);

 private:
    EdgeVector<Edge,
               DS_MST_BLOCK_SIZE,
               DS_MST_NO_OF_PAGES,
```

```
              DS_MST_PAGE_SIZE> _mst;

    EdgeWeightBig _totalWeight;
};


/** Adds an edge to the MST. */
inline void MST::add(const RelabeledEdge &edge)
{
    // The original source and target indices are restored ...
    Edge originalEdge( edge.originalSource(),
                       edge.originalTarget(),
                       edge.weight() );

    // ... and the original edge is added to the MST.
    add( originalEdge );
}


/**
   Adds an edge to the MST.
   This is an explicit copy of the add method for a RelabeledEdge
   in order to avoid virtual methods.
*/
inline void MST::add(const RelabeledEdgeWithoutSource &edge)
{
    // The original source and target indices are restored ...
    Edge originalEdge( edge.originalSource(),
                       edge.originalTarget(),
                       edge.weight() );

    // ... and the original edge is added to the MST.
    add( originalEdge );
}


/** Adds an edge to the MST. */
inline void MST::add(const Edge &edge)
{
    // add the given edge to the MST
    _mst.push_back( edge );

    // update total weight
    _totalWeight += edge.weight();
}


#endif // MST_H
```

B.2.6 sparingStack.h

```
#ifndef SPARINGSTACK_H
#define SPARINGSTACK_H

/**
   A CommonPoolOfBlocks manages blocks which can contain
   several elements of a specified value_type.
   It can be used by several SparingStacks.
*/
template <typename value_type, int elementsPerBlock, int noOfBlocks>
class CommonPoolOfBlocks
{
 public:
```

```
/**
   A block which can contain several elements.
*/
class Block
{
 public:
    /** The default constructor. */
    Block()
        : _size( 0 ), _prev( 0 )
        {}

    /**
       Adds a element to the block.
       Precondition(!): full() must return false.
    */
    void push(const value_type &element) {
        _elements[_size++] = element;
    }

    /**
       Returns a reference to the last element.
       Precondition(!): empty() must return false.
    */
    const value_type & top() const {return _elements[_size-1];}

    /**
       Removes the last element.
       Precondition(!): empty() must return false.
    */
    void pop() {_size--;}

    /**
       Removes all elements so that the block is empty.
    */
    void clear() {_size = 0;}

    /** Returns true iff the block contains no elements. */
    bool empty() const {return (_size==0);}

    /** Returns true iff the block can't adopt more elements. */
    bool full() const {return (_size==elementsPerBlock);}

    /** Returns the number of elements which are stored in the block. */
    int size() const {return _size;}

    /** Returns a reference to the element which is stored at the given position. */
    const value_type & operator[](int index) const {return _elements[index];}

    /** Sets the pointer to the previous block. */
    void setPrevBlock(Block *const prev) {_prev = prev;}

    /** Returns a pointer to the previous block. */
    Block * prevBlock() const {return _prev;}

 private:
    /** The number of elements which are stored in the block. */
    int _size;

    /** A pointer to the previous block. */
    Block *_prev;

    /** The elements. */
    value_type _elements[elementsPerBlock];
};


/** The default constructor. */
```

```
CommonPoolOfBlocks()
    : _free( &_blocks[noOfBlocks-1] ), _reserveMemory( 0 ),
      _totalNoOfBlocks( noOfBlocks ), _noOfFreeBlocks( noOfBlocks )
    {
        // initialize the linked list of free blocks
        for (int i=noOfBlocks-1; i>0; i--)
            _blocks[i].setPrevBlock(&_blocks[i-1]);
    }

/**
   The destructor.
   Blocks which have been created additionally are deleted.
*/
~CommonPoolOfBlocks() {
    for (int i=0; i<_additionalBlocks.size(); i++) {
        delete[] _additionalBlocks[i];
        DS_VERBOSE1( std::cout << std::endl
                     << "CommonPoolOfBlocks::~CommonPoolOfBlocks(): "
                     << "additionally created blocks deleted." << std::endl );
    }
}

/** Returns a pointer to a free block. */
Block * request() {

    if ( ! _free ) { // If there are no more free blocks...
        // check if reserve memory is available
        int noOfNewBlocks = _reserveMemory / sizeof( Block );
        if (noOfNewBlocks == 0) {
            // If there is not enough reserve memory, the program is aborted !
            std::cerr << "CommonPoolOfBlocks: request(): No free blocks available !"
                      << std::endl;
            abort();
        }

        DS_VERBOSE1( std::cout << std::endl << "CommonPoolOfBlocks::request(): "
                     << noOfNewBlocks << " new blocks created." << std::endl );

        // allocate new blocks according to the available reserve memory
        _reserveMemory -= noOfNewBlocks * sizeof( Block );

        DS_LOGGING2( _totalNoOfBlocks += noOfNewBlocks;
                     _noOfFreeBlocks += noOfNewBlocks );

        Block *newBlocks = new Block[noOfNewBlocks];
        _additionalBlocks.push_back( newBlocks );
        for (int i=0; i<noOfNewBlocks; i++) {
            newBlocks[i].setPrevBlock( _free );
            _free = &newBlocks[i];
        }
    }

    DS_LOGGING2( _noOfFreeBlocks-- );

    // return a free block
    Block *block = _free;
    _free = _free->prevBlock();
    return block;
}

/** Adds a block which is no longer used to the free-list. */
void release(Block *block) {
    block->setPrevBlock(_free);
    _free = block;

    DS_LOGGING2( _noOfFreeBlocks++ );
}
```

```
    /**
       Informs that more reserve memory is available.
       @param newMemory the space in bytes which can be used by the
                        request method if more blocks are required.
    */
    void increaseReserveMemory(int newMemory) {
        _reserveMemory += newMemory;
    }


    DS_LOGGING2(
        /** Returns the number of used blocks. */
        int noOfBlocksUsed() const {return _totalNoOfBlocks - _noOfFreeBlocks;}

        /** Returns the number of free blocks. */
        int noOfBlocksFree() const {return _noOfFreeBlocks;}
    )

 private:
    /** The blocks which are initially created. */
    Block _blocks[noOfBlocks];

    /** A pointer to the linked list of free blocks. */
    Block *_free;

    /**
       The reserve memory in bytes which can be used by the
       request method if more blocks are required.
    */
    int _reserveMemory;

    std::vector<Block*> _additionalBlocks;

    /**
       The total number of blocks. This is the 'noOfBlocks' plus the
       number of blocks which are created additionally using the
       reserve memory.
    */
    int _totalNoOfBlocks;

    /** The number of free blocks. */
    int _noOfFreeBlocks;

};


/**
   A stack which stores its elements in blocks which are managed by
   a CommonPoolOfBlocks.
*/
template <typename value_type, int elementsPerBlock, int noOfBlocks>
class SparingStack
{
 protected:
    /** The type of the used CommonPoolOfBlocks. */
    typedef CommonPoolOfBlocks<value_type, elementsPerBlock, noOfBlocks> Pool;
    /** The type of a Block which stores several elements. */
    typedef Pool::Block Block;

 public:
    /** The default constructor. */
    SparingStack() : _top( &_bottom ) {}

    /** The destructor. */
    ~SparingStack() {
        while (_top != &_bottom) {
```

```
            Block *oldBlock = _top;
            _top = _top->prevBlock();
            oldBlock->clear();
            _pool->release(oldBlock);
        }
    }

    /**
       Sets the CommonPoolOfBlocks which should be used.
       This method must be called after the object has been created and
       before it is used.
    */
    void setPool(Pool *pool) {_pool = pool;}

    /** Returns true iff the stack is empty. */
    bool empty() const {return _top->empty();}

    /** Adds an element to the stack. */
    void push(const value_type &element) {
        if ( _top->full() ) {
            // If the current block is full, request a new one.
            Block *newBlock = _pool->request();
            newBlock->setPrevBlock(_top);
            _top = newBlock;
        }
        _top->push(element);
    }

    /** Returns a reference to the last element. */
    const value_type & top() const {return _top->top();}

    /** Removes the last element. */
    void pop() {
        _top->pop();
        if ( (_top->empty()) && (_top != &_bottom) ) {
            // If the current block is now empty, release it.
            // (The 'bottom' block belongs to the stack and is never released.)
            Block *oldBlock = _top;
            _top = _top->prevBlock();
            _pool->release(oldBlock);
        }
    }

    /** Returns the number of elements which are stored in the stack. */
    int size() const {
        Block *current = _top;
        int blockCounter = 0;
        while (current != &_bottom) {
            blockCounter++;
            current = current->prevBlock();
        }
        return _top->size() + (blockCounter * DS_INT_EDGES_PER_BLOCK);
    }

 protected:
    /** A pointer to the used CommonPoolOfBlocks. */
    Pool *_pool;

    /** A pointer to the list of blocks which store the elements. */
    Block *_top;

    /** The bottom block which belongs to the stack. */
    Block _bottom;
};


/**
```

```
   A specialized SparingStack which stores RelabeledEdgeWithoutSource-objects.
*/
template <int elementsPerBlock, int noOfBlocks>
class REWS_SparingStack : public SparingStack<RelabeledEdgeWithoutSource,
                                              elementsPerBlock,
                                              noOfBlocks>
{
 public:
    /**
       Determines the edge with minimum weight.
       It is added to the resulting minimum spanning tree and
       the target node ID is returned.
       @param result a reference to the MST object which stores
                     the resulting minimum spanning tree
    */
    NodeID determineMinEdge(MST &result) const {
        // determine the edge with minimum weight
        RelabeledEdgeWithoutSource minEdge = top();
        Block *current = _top;
        while (true) {
            for (int index=current->size()-1; index >= 0; index--) {
                if ( (*current)[index] < minEdge ) minEdge = (*current)[index];
            }
            if (current == &_bottom) break;
            current = current->prevBlock();
        }

        // Add the edge with minimum weight incident to the
        // current node to the resulting minimum spanning tree.
        result.add( minEdge );

        DS_VERBOSE2( std::cout << "( !!! MST::add = "
                     << minEdge << " )" );

        return minEdge.target();
    }
};


#endif // SPARINGSTACK_H
```

B.2.7 duplicatesRemover.h

```
#ifndef DUPLICATESREMOVER_H
#define DUPLICATESREMOVER_H

/**
   A container which can be used as an 'intermediate station' in order
   to remove multiple edges. If there is more than one edge with the
   same target node (the source node is always the same during one pass),
   only the edge with minimum weight is preserved.
   If the capacity of the container is exhausted, further edges which can't
   be stored are output directly.
*/
template <typename SuperContainer>
class DuplicatesRemover
{
 public:
    /**
       Creates a DuplicatesRemover.
       @param superContainer a pointer to the super container, i.e. the container
                             that uses this DuplicatesRemover (either a Buckets or
                             a PQueue object). This is required in order to be able
                             to put the edges to the super container when the capacity
```

```
                        of this container is exhausted or when this container is
                        cleared.
    */
    DuplicatesRemover(SuperContainer *superContainer)
        : _superContainer( superContainer ), _size( 0 )
        {}

    /**
       Adds an edge to the container.
       If an edge with the same target node is already in the container,
       the new edge replaces the old one if it has a lower weight, otherwise
       it is discarded.
    */
    void insert(const RelabeledEdgeWithoutSource &edge, NodeID source) {
        int index = find( edge.target() );
        if ( empty(index) ) {
            // There is NO edge with the same target node.
            if (_size == _maxSize) {
                // The container is full, so the current edge has to be
                // output directly.
                _superContainer->add( RelabeledEdge(edge, source) );
            }
            else {
                // Add the edge to the container
                _hashMap[index] = HashMapElement( edge, _size );
                _edges[_size++] = edge;
            }
        }
        else {
            // There is another edge with the same target node.
            // So this a duplicate !
            DS_LOGGING2( logging.duplicates().increase() );
            if (edge < _hashMap[index].first) {
                // The current edge has a lower weight than the existing edge.
                // Therefore the existing edge is replaced.
                _edges[ _hashMap[index].second ] = edge;
                _hashMap[index].first = edge;
            }
        }
    }

    /** Clears the container. The edges are written to the appropriate buckets. */
    void clear(NodeID source) {
        while ( _size > 0 ) {
            clearLocal( _edges[--_size].target() );
            _superContainer->add( RelabeledEdge(_edges[_size], source) );
        }
    }

private:
    /** The capacity of the container. */
    static const EdgeCount _maxSize = 1024;

    /** The size of the hash map. */
    static const EdgeCount _hashMapSize = 2 * _maxSize;

    /**
       The type of a hashMap-element.
       The first component stores the edge, the second the index in the
       array '_edges'.
    */
    typedef std::pair<RelabeledEdgeWithoutSource, int> HashMapElement;

    /**
       This array contains all edges without vacancies.
       This is useful in order to clear the container without traversing
       the whole hashMap.
```

```
    */
    RelabeledEdgeWithoutSource _edges[_maxSize];

    /** The hashMap which contains the edges. */
    HashMapElement _hashMap[_hashMapSize];

    /** The number of edges which are stored in this container. */
    EdgeCount _size;

    /**
       A pointer to the super container, i.e. the container that uses
       this DuplicatesRemover (either a Buckets or a PQueue object).
       This is required in order to be able to put the edges to the
       super container when the capacity of this container is exhausted
       or when this container is cleared.
    */
    SuperContainer *_superContainer;

    /** Returns the hash value for the given node ID. */
    int hashFunction(NodeID x) const {
        return x % _hashMapSize;
    }

    /**
       Returns the position of an edge with the given node ID in the hashMap.
       If no appropriate edge is found, the position where such an edge should
       be stored is returned.
    */
    int find(NodeID x) const {
        int i = hashFunction(x);
        while ((_hashMap[i].first.target() != x) && ( ! empty(i) ))
            i = (i+1) % _hashMapSize;
        return i;
    }

    /**
       Removes entries from the hashMap beginning with the hash value of
       the given node ID and ending with the first empty entry which is found.
    */
    void clearLocal(NodeID x) {
        int i = hashFunction(x);
        while ( ! empty(i) ) {
            _hashMap[i].first = RelabeledEdgeWithoutSource();
            i = (i+1) % _hashMapSize;
        }
    }

    /** Returns true iff the given position in the hashMap is empty. */
    bool empty(int index) const {
        if ( (_hashMap[index].first.originalSource() == 0) &&
             (_hashMap[index].first.originalTarget() == 0) ) return true;
        return false;
    }

};

#endif // DUPLICATESREMOVER_H
```

B.3 Base Case

B.3.1 kruskal.h

```
#ifndef KRUSKAL_H
#define KRUSKAL_H

#include <vector>

#include "../data_structures/edgeVector.h"
#include "../data_structures/mst.h"

/**
   Represents Kruskal's algorithm for determining a Minimum Spanning Tree
   of a weighted graph and the required union/find data structure.
*/
template < typename EdgeType = Edge >
class Kruskal
{
 public:
    /**
       Computes a MST of a graph.
       @param graph a reference to an EdgeVector which represents the graph
       @param result a reference to a MST object which stores the result
    */
    Kruskal(EdgeVector<EdgeType> &graph, MST &result)
        : _graph( graph ), _result( result )
        {
            computeMST();
        }

    /**
       The destructor.
       The given graph is deleted.
    */
    ~Kruskal() {delete &_graph;}

 private:
    /** Computes a MST of the graph. */
    void computeMST();

    /** Initializes the union/find data structure. */
    void initUnionFind();


    /**
       Performs a find operation.
       Path compression is applied.
       @param node the identifier of the node whose set should be determined
       @return the identifier of the canonical node which represents the set
               which "node" belongs to
    */
    NodeID find(NodeID node);

    /**
       Performs a union operation.
       @return true iff node1 and node2 have belonged to different sets
    */
    bool unite(NodeID node1, NodeID node2);


    /** Reference to the given graph. */
    EdgeVector<EdgeType> &_graph;

    /** Reference to a MST object which stores the result. */
    MST &_result;
```

```
    /**
       A vector which contains for each node the identifier of the parent node.
       The canonical node of a set (= the root of a tree) points to itself.
    */
    std::vector<NodeID> _parent;

    /**
       A vector which contains for each node the height of the belonging tree.
       Only the values of canonical nodes are relevant.
    */
    std::vector<char> _height;

    /** Counts the number of edges which have been added to the resulting MST */
    EdgeCount edgesAddedToResult;

};


/**
   Performs a find operation.
   Path compression is applied.
   @param node the identifier of the node whose set should be determined
   @return the identifier of the canonical node which represents the set
           which "node" belongs to
*/
template < typename EdgeType >
inline NodeID Kruskal<EdgeType>::find(NodeID node)
{
    // find the root
    NodeID root = node;
    while (_parent[root] != root) root = _parent[root];

    // apply path compression
    NodeID tmp;
    while (_parent[node] != node) {
        tmp = _parent[node];
        _parent[node] = root;
        node = tmp;
    }

    return root;
}

/**
   Performs a union operation.
   @return true iff node1 and node2 have belonged to different sets
*/
template < typename EdgeType >
inline bool Kruskal<EdgeType>::unite(NodeID node1, NodeID node2)
{
    NodeID root1 = find(node1);
    NodeID root2 = find(node2);

    // A cycle would be produced. Therefore the union operation is cancelled.
    if (root1 == root2) return false;

    // Add the smaller tree to the bigger tree
    if (_height[root1] < _height[root2]) {
        _parent[root1] = root2;
    }
    else {
        _parent[root2] = root1;

        // Increment the height of the resulting tree if both trees have
        // the same height
        if (_height[root1] == _height[root2]) _height[root1]++;
```

```
    }
    return true;
}


#endif // KRUSKAL_H
```

B.3.2 kruskal.cpp

```
#include "kruskal.h"

#include "../utils/utils.h"

/** Computes a MST of the graph. */
template < typename EdgeType >
void Kruskal<EdgeType>::computeMST()
{
    // sort
    DS_LOGGING2( logMemoryUsage("before sorting") );
    DS_VERBOSE1( std::cout << std::endl << "Kruskal: sort edges" << std::endl );
    _graph.sortByWeight();
    DS_LOGGING2( logMemoryUsage("after sorting") );
    DS_LOGGING1( logging.events().push_back(LoggingEvent("Kruskal: edges sorted")) );



    // initialize
    DS_VERBOSE1( std::cout << std::endl << "Kruskal: initialize data structures"
                           << std::endl );
    initUnionFind();
    DS_LOGGING2( logMemoryUsage("after initialization") );
    DS_LOGGING1( logging.events().push_back(
                    LoggingEvent("Kruskal: data structures initialized")) );



    // process edges
    DS_VERBOSE1( std::cout << std::endl << "Kruskal: process edges" << std::endl;
                 Percent percent(_graph.noOfEdges()) );
    // abort the for-loop when the MST is complete or when all edges have been processed.
    for (EdgeCount i=0;
         (edgesAddedToResult < _graph.noOfNodes()-1) && (i<_graph.noOfEdges());
         i++) {
        DS_VERBOSE1( percent.printStatus(i) );

        // perform union operation according to the current edge
        if (unite( _graph[i].source(), _graph[i].target() )) {
            // The current edge doesn't produce a cyle.
            // Hence, add it to the resulting MST.
            _result.add(_graph[i]);
            edgesAddedToResult++;
        }
    }

    DS_LOGGING1( logging.events().push_back(LoggingEvent("Kruskal: edges processed")) );

}

/** Initializes the union/find data structure. */
template < typename EdgeType >
void Kruskal<EdgeType>::initUnionFind()
{
    // init parent vector
```

```
    _parent.resize(_graph.noOfNodes());
    for (NodeCount i=0; i<_parent.size(); i++) _parent[i] = i;

    // init height vector
    _height.resize(_graph.noOfNodes());

    edgesAddedToResult = 0;
}
```

B.4 Buckets

B.4.1 buckets.h

```
#ifndef BUCKETS_H
#define BUCKETS_H

#include <vector>

#include "../stxxl/containers/stack.h"
#include "../data_structures/sparingStack.h"

#include "../data_structures/edgeVector.h"
#include "../data_structures/mst.h"


/**
   Represents an algorithm for reducing the number of nodes
   of a graph (in order to compute a minimum spanning tree)
   and the required data structures.
   This implementation uses several buckets.

   template parameter:
   The number of nodes per bucket.
   This value doesn't apply to the first bucket, because
   it contains the edges of the first "_noOfNodesInIntMem"
   nodes.
*/
template <NodeCount nodesPerBucket = 1>
class Buckets
{
public:
    typedef int BucketID;

private:
    /** The type of an external bucket which contains the edges of several nodes. */
    typedef stxxl::STACK_GENERATOR<RelabeledEdge,
                                   stxxl::external,
                                   stxxl::grow_shrink2,
                                   DS_EXT_PAGE_SIZE,
                                   DS_EXT_BLOCK_SIZE>::result EdgesOfSeveralNodes;

    /** The type of an internal bucket which contains the edges of one node. */
    typedef REWS_SparingStack<DS_INT_EDGES_PER_BLOCK, DS_INT_NO_OF_BLOCKS> EdgesOfOneNode;

    /** The type of the common pool of the internal buckets. */
    typedef CommonPoolOfBlocks<RelabeledEdgeWithoutSource,
                               DS_INT_EDGES_PER_BLOCK,
                               DS_INT_NO_OF_BLOCKS> PoolEdgesOfOneNode;

public:
    /** Computes the number of external buckets that are needed. */
    static BucketID noOfExtBuckets(NodeCount noOfNodes, NodeCount noOfNodesInIntMem);
```

```
    /**
       Reduces the number of nodes of a graph (in order to
       compute a minimum spanning tree).
       @param graph a reference to an EdgeVector which represents the graph
       @param result a reference to a MST object which stores the result
       @param noOfBuckets the number of external buckets which should be used
       @param noOfNodesInIntMem the number of nodes which fit in internal memory
    */
    Buckets(EdgeVector<Edge> &graph, MST &result,
            BucketID noOfBuckets, NodeCount noOfNodesInIntMem)
        : _graph( graph ), _result( result ), _extBuckets( noOfBuckets-1 ),
          _noOfNodesInIntMem( noOfNodesInIntMem ),
          _prefetchPool( DS_EXT_PREFETCH_POOL ), _writePool( DS_EXT_WRITE_POOL )
        {
            initBuckets();
            reduceNodes();
        }

    /**
       Returns a pointer to the first external bucket which contains the
       edges of the nodes which fit in internal memory.
    */
    InternalMemoryBucket* getIntMemBucket() {return _firstExtBucket;}

    /**
       Adds the given edge to the appropriate external bucket or
       to the appropriate internal bucket if the edge belongs
       to the external bucket which is processed at the moment.
    */
    void add(const RelabeledEdge &edge);

private:
    /** Reference to the given graph. */
    EdgeVector<Edge> &_graph;

    /** Reference to a MST object which stores the result. */
    MST &_result;

    /** The number of nodes which fit in internal memory. */
    NodeCount _noOfNodesInIntMem;

    /**
       The first external bucket which contains the
       edges of the nodes which fit in internal memory.
    */
    InternalMemoryBucket *_firstExtBucket;

    /**
       The external buckets.
       Each bucket contains the edges of several nodes.
    */
    std::vector<EdgesOfSeveralNodes*> _extBuckets;

    /** The prefetch pool that is used by the external buckets. */
    stxxl::prefetch_pool<EdgesOfSeveralNodes::block_type> _prefetchPool;

    /** The write pool that is used by the external buckets. */
    stxxl::write_pool<EdgesOfSeveralNodes::block_type> _writePool;

    /**
       The internal buckets.
       Each bucket contains the edges of one node.
    */
    EdgesOfOneNode _intBuckets[nodesPerBucket];
```

```
    /**
       The identifier of the first node in the current external bucket will be used
       to compute the internal bucket index of a node.
    */
    NodeID _firstNodeIDofCurrentBucket;

    /**
       Initializes the external buckets and distributes the edges to the
       external buckets.
    */
    void initBuckets();

    /**
       Reduces the number of nodes.
       Only the first bucket "survives".
    */
    void reduceNodes();

    /**
       Returns the ID of the bucket which contains the edges
       of the given node.
       The first external bucket has the ID -1 because it is a
       special case. The second bucket is the first element
       of _extBuckets and has the ID 0 and so on.
    */
    BucketID bucketID(NodeID nodeID) const;

    /** Adds the given edge to the appropriate external bucket. */
    void addToExternalBucket(const RelabeledEdge &edge);

    /** Adds the given edge to the bucket which is specified by newBucketID. */
    void addToExternalBucket(const RelabeledEdge &edge, BucketID newBucketID);
};


/**
   Returns the ID of the bucket which contains the edges
   of the given node.
   The first external bucket has the ID -1 because it is a
   special case. The second bucket is the first element
   of _extBuckets and has the ID 0 and so on.
*/
template <NodeCount nodesPerBucket>
inline Buckets<nodesPerBucket>::BucketID Buckets<nodesPerBucket>::bucketID(NodeID nodeID)
const
{
    if (nodeID < _noOfNodesInIntMem) return -1;
    return (nodeID - _noOfNodesInIntMem) / nodesPerBucket;
}

/** Adds the given edge to the appropriate external bucket. */
template <NodeCount nodesPerBucket>
inline void Buckets<nodesPerBucket>::addToExternalBucket(const RelabeledEdge &edge)
{
    if ( edge.isSelfLoop() ) return; // Self loops can be ignored.
    addToExternalBucket( edge, bucketID(edge.source()) );
}

/** Adds the given edge to the bucket which is specified by newBucketID. */
template <NodeCount nodesPerBucket>
inline void Buckets<nodesPerBucket>::addToExternalBucket(const RelabeledEdge &edge,
                                                         BucketID newBucketID)
{
    if (newBucketID == -1) {
        // special case
        _firstExtBucket->push_back( edge );
    }
```

```
    else {
        _extBuckets[newBucketID]->push( edge );
    }
}

/**
   Adds the given edge to the appropriate external bucket or
   to the appropriate internal bucket if the edge belongs
   to the external bucket which is processed at the moment.
*/
template <NodeCount nodesPerBucket>
inline void Buckets<nodesPerBucket>::add(const RelabeledEdge &edge)
{
    if (edge.source() >= _firstNodeIDofCurrentBucket) {
        // The relabeled edge still belongs to the current external bucket.
        // Therefore it is added to the appropriate internal bucket of one
        // single node.
        _intBuckets[edge.source() - _firstNodeIDofCurrentBucket]
            .push(edge);
    }
    else {
        // The relabeled edge now belongs to another external bucket.
        // Therefore it is added to the appropriate external bucket.
        addToExternalBucket( edge, bucketID(edge.source()) );
    }
}

#endif // BUCKETS_H
```

B.4.2 buckets.cpp

```
#include "buckets.h"
#include "../data_structures/duplicatesRemover.h"
#include "../utils/randomizer.h"


/**
   Initializes the external buckets and distributes the edges
   to the external buckets.
*/
template <NodeCount nodesPerBucket>
void Buckets<nodesPerBucket>::initBuckets()
{
    // create the external buckets
    DS_VERBOSE1( std::cout << std::endl << "create external buckets" << std::endl;
                 Percent percent(_extBuckets.size()) );

    _firstExtBucket = new InternalMemoryBucket(_noOfNodesInIntMem,0);
    for (BucketID i=0; i<_extBuckets.size(); i++) {
        DS_VERBOSE1( percent.printStatus(i) );
        _extBuckets[i] = new EdgesOfSeveralNodes( _prefetchPool, _writePool, 0 );
    }


    // distribute the edges to the external buckets
    DS_RANDOMIZE( Randomizer<RandomizerFeistel> randomizer(_graph.noOfNodes()) );

    DS_VERBOSE1( std::cout << std::endl
                 << "distribute edges to external buckets" << std::endl;
                 percent.reinit(_graph.size(), true) );

    for (; !_graph.empty(); _graph.pop_back()) {
        DS_DONT_RANDOMIZE( RelabeledEdge edge(_graph.back()) );
        DS_RANDOMIZE( RelabeledEdge edge = randomizer.randomize(_graph.back()) );
```

```
        DS_VERBOSE1( percent.printStatus(_graph.size()) );

        DS_VERBOSE2( std::cout << edge; )


        // The source has to be greater than the target.
        if (edge.source() < edge.target()) edge.swap();

        DS_VERBOSE2( std::cout << "  swap  " << edge
                     << " bucketID: " << bucketID(edge.source())
                     << "  relabeled " );

        // add the current edge to the appropriate external bucket
        addToExternalBucket(edge);


        DS_VERBOSE2(
            if (bucketID(edge.source()) == -1)
                std::cout << _firstExtBucket->back() << std::endl;
            else
                std::cout << _extBuckets[bucketID(edge.source())]->top() << std::endl; )
    }

    // delete the input graph
    delete &_graph;

    DS_LOGGING1( logging.events().push_back(
                     LoggingEvent("edges distributed to external buckets")) );
}


/**
   Reduces the number of nodes.
   Only the first bucket "survives".
*/
template <NodeCount nodesPerBucket>
void Buckets<nodesPerBucket>::reduceNodes()
{
    // create/initialize the internal buckets
    DS_REMOVE_DUPLICATES(
        DuplicatesRemover< Buckets<nodesPerBucket> > *duplRem =
            new DuplicatesRemover< Buckets<nodesPerBucket> >(this);
    )

    PoolEdgesOfOneNode *pool = new PoolEdgesOfOneNode();
    for (NodeCount i=0; i<nodesPerBucket; i++) {
        _intBuckets[i].setPool(pool);
    }

    // The identifier of the first node in the current external bucket will be used
    // to compute the internal bucket index of a node.
    _firstNodeIDofCurrentBucket = _noOfNodesInIntMem
        + ( _extBuckets.size() * nodesPerBucket );

    // Process all buckets from the last to the first but one.
    for (BucketID currentExtBucketID=_extBuckets.size()-1; !_extBuckets.empty();
         currentExtBucketID--) {

        _firstNodeIDofCurrentBucket -= nodesPerBucket;

        EdgesOfSeveralNodes & currentExtBucket = *_extBuckets[currentExtBucketID];

        // Set the size of the prefetch buffer that is used by the current external bucket.
        currentExtBucket.set_prefetch_aggr( DS_EXT_PREFETCH_POOL );

        DS_LOGGING2( logging.buckets().push_back(
```

```
                LoggingBucket(currentExtBucketID, nodesPerBucket,
                              currentExtBucket.size())) );

// Read current external bucket and distribute the edges to the internal buckets.
DS_VERBOSE1( std::cout << std::endl
             << "process bucket " << currentExtBucketID << std::endl
             << "  read    ";
             Percent percent(currentExtBucket.size(), true, 20) );

for (; ! currentExtBucket.empty(); currentExtBucket.pop() ) {
    DS_VERBOSE1( percent.printStatus(currentExtBucket.size()) );
    _intBuckets[currentExtBucket.top().source() - _firstNodeIDofCurrentBucket]
        .push(currentExtBucket.top());
}


DS_LOGGING2( logging.buckets().back().setBlocksUsed(pool->noOfBlocksUsed());
             logging.buckets().back().setBlocksFree(pool->noOfBlocksFree()) );

// Relabel the edges. All nodes (in the current bucket)
// are processed from the last to the first.
DS_VERBOSE1( std::cout << std::endl << "  relabel ";
             percent.reinit(nodesPerBucket, true, 20) );

for (NodeID currentNodeIndex = nodesPerBucket-1;
     currentNodeIndex < nodesPerBucket; // NodeID is unsigned
     currentNodeIndex--) {

    EdgesOfOneNode &currentIntBucket = _intBuckets[currentNodeIndex];

    DS_LOGGING2( logging.buckets().back().
                      addEdgesProcessed(currentIntBucket.size()) );
    DS_VERBOSE1( percent.printStatus(currentNodeIndex) );
    DS_VERBOSE2( std::cout << std::endl
                 << "** currentNodeIndex = " << currentNodeIndex );

    if ( currentIntBucket.empty() ) continue;

    // Determine the edge with minimum weight incident to the current node
    // and add it to the resulting minimum spanning tree.
    NodeID newSource = currentIntBucket.determineMinEdge(_result);

    // Relabel all edges of the current node.
    for(; ! currentIntBucket.empty(); currentIntBucket.pop() ) {

        // Relabel the current edge.
        // Self loops can be ignored.
        if ( currentIntBucket.top().target() != newSource ) {

            DS_DONT_REMOVE_DUPLICATES(
                add( RelabeledEdge(currentIntBucket.top(), newSource) )
                );

            DS_REMOVE_DUPLICATES(
                duplRem->insert( currentIntBucket.top(), newSource )
                );

        }
    }

    DS_REMOVE_DUPLICATES( duplRem->clear(newSource) );

}

DS_LOGGING2(
    system("ps -C mst -o rss,%mem --no-headers | head -n 1 >> memoryLog.tmp") );
```

```
        // Delete the current bucket which is now empty.
        delete _extBuckets[currentExtBucketID];
        _extBuckets.pop_back();

        // The released memory of the current external bucket
        // can be used by the internal buckets.
        pool->increaseReserveMemory( DS_EXT_BLOCK_SIZE * DS_EXT_PAGE_SIZE );

    }

    DS_LOGGING2( logging.buckets().push_back(
                    LoggingBucket(-1, _firstExtBucket->noOfNodes(),
                                  _firstExtBucket->noOfEdges())) );

    // Delete the common pool of the internal buckets.
    delete pool;
    DS_REMOVE_DUPLICATES( delete duplRem );
}


/** Computes the number of external buckets that are needed. */
template <NodeCount nodesPerBucket>
Buckets<nodesPerBucket>::BucketID
Buckets<nodesPerBucket>::noOfExtBuckets(NodeCount noOfNodes, NodeCount noOfNodesInIntMem) {
    if (noOfNodes <= noOfNodesInIntMem) return 0;

    NodeCount noOfNodesInExtMem = noOfNodes - noOfNodesInIntMem;
    BucketID noOfBuckets = noOfNodesInExtMem / nodesPerBucket;
    if (noOfNodesInExtMem % nodesPerBucket != 0) noOfBuckets++; // round up
    noOfBuckets++; // first external bucket

    return noOfBuckets;
}
```

B.5 Priority Queue

B.5.1 pQueue.h

```
#ifndef PQUEUE_H
#define PQUEUE_H


#include "../stxxl/containers/priority_queue.h"

#include "../data_structures/edgeVector.h"
#include "../data_structures/mst.h"


/**
   Represents an algorithm for reducing the number of nodes
   of a graph (in order to compute a minimum spanning tree)
   and the required data structures.
   This implementation uses an external priority queue.
*/
class PQueue
{
 private:
    /** The type of the priority queue that is used during the node reduction phase. */
    typedef stxxl::PRIORITY_QUEUE_GENERATOR<RelabeledEdge,
                                            SourceWeightOrdering<RelabeledEdge>,
                                            DS_PQUEUE_INTERNAL_MEMORY,
                                            DS_PQUEUE_MAX_SIZE>::result PriorityQueue;
```

```
public:
    /**
       Reduces the number of nodes of a graph (in order to
       compute a minimum spanning tree).
       @param graph a reference to an EdgeVector which represents the graph
       @param result a reference to a MST object which stores the result
       @param noOfNodesInIntMem the number of nodes which fit in internal memory
    */
    PQueue(EdgeVector<Edge> &graph, MST &result, NodeCount noOfNodesInIntMem)
        : _graph( graph ), _result( result ),
          _noOfNodesInIntMem( noOfNodesInIntMem ),
          _prefetchPool( DS_PQUEUE_PREFETCH_POOL ), _writePool( DS_PQUEUE_WRITE_POOL ),
          _pqueue(_prefetchPool, _writePool)
        {
            initPQueue();
            reduceNodes();
        }

    /**
       Returns a pointer to the first external bucket which contains the
       edges of the nodes which fit in internal memory.
    */
    InternalMemoryBucket* getIntMemBucket() {return _firstExtBucket;}

    /**
       Adds the given edge to the first external bucket or to the
       priority queue depending on the source vertex ID.
    */
    void add(const RelabeledEdge &edge);

private:
    /** Reference to the given graph. */
    EdgeVector<Edge> &_graph;

    /** Reference to a MST object which stores the result. */
    MST &_result;

    /** The number of nodes which fit in internal memory. */
    NodeCount _noOfNodesInIntMem;

    /**
       The first external bucket which contains the
       edges of the nodes which fit in internal memory.
    */
    InternalMemoryBucket *_firstExtBucket;

    /** The prefetch pool that is used by the priority queue. */
    stxxl::prefetch_pool<PriorityQueue::block_type> _prefetchPool;

    /** The write pool that is used by the priority queue. */
    stxxl::write_pool<PriorityQueue::block_type> _writePool;

    /** The priority queue. */
    PriorityQueue _pqueue;

    /**
       Initializes the priority queue and distributes the edges to the
       first external bucket and the priority queue.
    */
    void initPQueue();

    /**
       Reduces the number of nodes.
       Only the first external bucket "survives".
    */
    void reduceNodes();
```

```
};


/**
   Adds the given edge to the first external bucket or to the
   priority queue depending on the source vertex ID.
*/
inline void PQueue::add(const RelabeledEdge &edge)
{
    if ( edge.isSelfLoop() ) return; // Self loops can be ignored.

    if (edge.source() < _noOfNodesInIntMem) {
        // add to the first external bucket
        _firstExtBucket->push_back( edge );
    }
    else {
        // add to the priority queue
        _pqueue.push( edge );
    }
}


#endif // PQUEUE_H
```

B.5.2 pQueue.cpp

```
#include "pQueue.h"
#include "../data_structures/duplicatesRemover.h"
#include "../utils/randomizer.h"


/**
   Initializes the priority queue and distributes the edges to the
   first external bucket and the priority queue.
*/
void PQueue::initPQueue()
{
    // create the first external bucket
    _firstExtBucket = new InternalMemoryBucket(_noOfNodesInIntMem,0);


    // distribute the edges to the first external bucket and the priority queue
    DS_RANDOMIZE( Randomizer<RandomizerFeistel> randomizer(_graph.noOfNodes()) );

    DS_VERBOSE1( std::cout << std::endl
                 << "distribute edges to the first external bucket and the priority queue"
                 << std::endl;
                 Percent percent(_graph.size(), true) );

    for (; !_graph.empty(); _graph.pop_back()) {
        DS_DONT_RANDOMIZE( RelabeledEdge edge(_graph.back()) );
        DS_RANDOMIZE( RelabeledEdge edge = randomizer.randomize(_graph.back()) );

        DS_VERBOSE1( percent.printStatus(_graph.size()) );
        DS_VERBOSE2( std::cout << edge << std::endl );

        // The source has to be greater than the target.
        if (edge.source() < edge.target()) edge.swap();

        // add the current edge to the first external bucket
        // or to the priority queue
        add(edge);
    }
```

```
    // delete the input graph
    delete &_graph;

    DS_LOGGING1( logging.events().push_back(
                    LoggingEvent("edges distributed to external buckets")) );
}


/**
   Reduces the number of nodes.
   Only the first external bucket "survives".
*/
void PQueue::reduceNodes()
{
    if ( _pqueue.empty() ) return;

    DS_REMOVE_DUPLICATES(
        DuplicatesRemover<PQueue> *duplRem = new DuplicatesRemover<PQueue>(this) );

    // get shortest edge incident to the last node
    RelabeledEdge minWeightEdge( _pqueue.top() );
    _pqueue.pop();
    _result.add( minWeightEdge );

    DS_VERBOSE1( std::cout << std::endl << "reduce nodes" << std::endl;
                 Percent percent( minWeightEdge.source() - _noOfNodesInIntMem );
                 NodeCount processedNodes = 0 );

    // Process all edges in the priority queue
    while ( ! _pqueue.empty() ) {
        // get current edge
        RelabeledEdge currentEdge( _pqueue.top() );
        _pqueue.pop();

        // check whether the current edge has the same source vertex as the predecessor
        if ( minWeightEdge.source() == currentEdge.source() ) {
            // throw the old source vertex away
            RelabeledEdgeWithoutSource currentEdgeWithoutSource( currentEdge );

            DS_DONT_REMOVE_DUPLICATES(
                // add the relabeled edge to the first external bucket
                // resp. to the priority queue
                add( RelabeledEdge(currentEdgeWithoutSource, minWeightEdge.target()) );
            )

            DS_REMOVE_DUPLICATES(
                // add the edge to the DuplicateRemover
                duplRem->insert( currentEdgeWithoutSource, minWeightEdge.target() );

                // clear the DuplicateRemover if this is the last edge incident to the
                // current source vertex
                if ((_pqueue.empty()) ||
                    (_pqueue.top().source() != minWeightEdge.source())) {
                    duplRem->clear( minWeightEdge.target() );
                }
            )

        }
        else {
            // the current edge is the shortest one incident to the currently last node
            minWeightEdge = currentEdge;
            _result.add( minWeightEdge );

            DS_VERBOSE1( processedNodes++;
                         percent.printStatus(processedNodes) );
        }
```

```
    }

    DS_REMOVE_DUPLICATES( delete duplRem );
}
```

B.6 Utilities

B.6.1 generate.h

```
#ifndef GENERATE_H
#define GENERATE_H

#include "../data_structures/edgeVector.h"

/**
 Generates a complete graph with the given number of nodes.
 The weights of the edges are chosen randomly.
*/
EdgeVector<> * generateCompleteGraph(NodeCount noOfNodes);

/**
   Generates a random graph with the given number of nodes
   and the given number of edges.
   The weights of the edges are chosen randomly, too.
*/
EdgeVector<> * generateRandomGraph(NodeCount noOfNodes, EdgeCount noOfEdges);

/**
   Generates a grid graph with "noOfNodesX * noOfNodesY" nodes.
   The weights of the edges are chosen randomly.
*/
EdgeVector<> * generateGridGraph(NodeCount noOfNodesX, NodeCount noOfNodesY);

/**
   Generates a geometric graph.
   The given number of nodes is placed in a square and each node is connected with
   the given number of nearest neighbours. Duplicates (which come into existence when
   two nodes select each other) are removed.
*/
EdgeVector<> * generateGeometricGraph(NodeCount noOfNodes, NodeCount noOfNeighbours);

/** Returns a random node identifier between 0 and n-1. */
inline NodeID randomNodeID(NodeID n);

/** Returns a random edge weight. */
inline EdgeWeight randomEdgeWeight();

#endif // GENERATE_H
```

B.6.2 importExport.h

```
#ifndef IMPORTEXPORT_H
#define IMPORTEXPORT_H

#include <iostream>
```

```
#include "../data_structures/edgeVector.h"


/**
   Imports an EdgeVector from a file.
   expected format:
   the format which is expected by 'importEdgeVectorListOfEdges' or
   the format which is expected by 'importEdgeVectorAdjList' or
   the format which is expected by 'importEdgeVectorCompressed'
*/
EdgeVector<> * importEdgeVector(const std::string &filename);


/**
   Imports an EdgeVector from an istream.
   expected format:

   noOfNodes noOfEdges
   source0 target0 weight0
   source1 target1 weight1
   source2 target2 weight2
   ...

   sourceX and targetX are node indizes; the first used node index should be 0.
*/
EdgeVector<> * importEdgeVectorListOfEdges(std::istream &in);


/**
   Imports an EdgeVector from an istream.
   expected format:

   'a'
   noOfNodes
   adjacency list of node 1
   -1
   adjacency list of node 2
   -1
   ...

   Each adjacency list is a list of pairs: node1 weight1 node2 weight2 ...
   The first used node index should be 1.
*/
EdgeVector<> * importEdgeVectorAdjList(std::istream &in);


/**
   Imports an EdgeVector from a file.
   expected format:

   'c'
   noOfNodes noOfEdges
   source0 target0 weight0
   source1 target1 weight1
   source2 target2 weight2
   ...

   sourceX and targetX are node indizes; the first used node index should be 0.
   Each 32-bit number (sourceX, targetX, weightX) should be represented by 4 chars
   and there should be no spaces and newlines after the number of edges.
   If necessary, several files are used so that the file size limit isn't exceeded.
*/
EdgeVector<> * importEdgeVectorCompressed(const std::string &filename);


/**
```

```
   Exports an EdgeVector to an ostream.
   created format:

   'a'
   noOfNodes
   adjacency list of node 1
   -1
   adjacency list of node 2
   -1
   ...

   Each adjacency list is a list of pairs: node1 weight1 node2 weight2 ...
   The first used node index is 1.
*/
void exportEdgeVectorAdjList(std::ostream &out, EdgeVector<> &edges);


/**
   Exports an EdgeVector to a file.
   created format:

   'c'
   noOfNodes noOfEdges
   source0 target0 weight0
   source1 target1 weight1
   source2 target2 weight2
   ...

   sourceX and targetX are node indizes; the first used node index is 0.
   Each 32-bit number (sourceX, targetX, weightX) is represented by 4 chars
   and there are no spaces and newlines after the number of edges.
   If necessary, several files are used so that the file size limit isn't exceeded.
*/
void exportEdgeVectorCompressed(const std::string &filename, EdgeVector<> &edges);


/**
   Returns the number of nodes of the graph which is stored in the file
   with the given name.
*/
NodeCount noOfNodesInFile(const std::string &filename);


#endif // IMPORTEXPORT_H
```

B.6.3 randomizer.h

```
#ifndef RANDOMIZER_H
#define RANDOMIZER_H

/**
   A bijection that uses the linear congruential method.
*/
class RandomizerLinearCongruence
{
 public:
    /**
       The constructor.
       A bijection over {0,...p-1} is initialized, whereat p is the
       next prime number greater than or equal to a given n.
    */
    RandomizerLinearCongruence(NodeCount noOfNodes)
        {determineNextPrime(noOfNodes);}
```

```
    /** The bijection. */
    NodeID operator()(NodeID nodeID) const {
        typedef unsigned long long int bigNumber;

        bigNumber x = nodeID;
        const bigNumber a = 321321;
        const bigNumber c = 1;
        bigNumber m = _prime;

        // the linear congruential method
        return (a * x + c) % m;
    }

 private:
    /** The next prime number >= the given number of nodes. */
    NodeCount _prime;

    /** Determines the next prime number >= the given number of nodes. */
    void determineNextPrime(NodeCount noOfNodes) {
        NodeCount p;
        for (p = noOfNodes-1; ! isPrime(p); p++);
        _prime = p;
    }

    /** Returns true iff the given number is prime. */
    bool isPrime(NodeCount p) const {
        for (NodeCount i=2; i <= std::sqrt((double)p); i++) {
            if (p % i == 0) return false;
        }
        return true;
    }
};


/**
   A bijection that uses Feistel permutations.
*/
class RandomizerFeistel
{
 public:
    /**
       The constructor.
       A bijection over {0,...,r^2-1} is initialized, whereat r is the
       next integer greater than or equal to the square root of a given n.
    */
    RandomizerFeistel(NodeCount noOfNodes) {
        // compute the next integer >= the square root of the given number of nodes
        _sqRoot = (NodeID)std::sqrt((double)noOfNodes);
        if ((NodeCount)_sqRoot * _sqRoot < noOfNodes) _sqRoot++;
        // initialize the table of random numbers (used by the Feistel permutations)
        initRandomNumbers();
    }

    /** The bijection. */
    NodeID operator()(NodeID nodeID) const {
        // Split the given number into two parts
        int a = nodeID / _sqRoot;
        int b = nodeID % _sqRoot;

        // Perform the Feistel permutations
        for (int i=0; i<_noOfIterations; i++) {
            int newA = b;
            b = a + randomNumbers[i][b];
            if (b >= _sqRoot) b -= _sqRoot; // mod _sqRoot
            a = newA;
        }
```

```
        // Recombine a and b to obtain the result
        return a * _sqRoot + b;
    }

 private:
    /** The number of performed Feistel permutations. */
    static const int _noOfIterations = 2;

    /** The maximum size of _sqRoot. */
    static const int _maxSqRoot = 0x10000;

    /** The next integer >= the square root of the given number of nodes. */
    NodeID _sqRoot;

    /** The table of random numbers (used by the Feistel permutations). */
    int randomNumbers[_noOfIterations][_maxSqRoot];

    /** Initializes the table of random numbers. */
    void initRandomNumbers() {
        for (int i=0; i<_noOfIterations; i++)
            for (int j=0; j<_maxSqRoot; j++) {
                // each random number is an integer in {0,...,_sqRoot-1}
                randomNumbers[i][j] = (int)(rand() / (double)(RAND_MAX+1.0) * _sqRoot);
            }
    }
};


/**
   A bijection over {0,...,n-1} that can be used to randomize an Edge
   in order to obtain a (pseudo-)random permutation.
   Either RandomizerLinearCongruence or RandomizerFeistel can be used
   as underlying bijection.
*/
template <typename Bijection = RandomizerLinearCongruence>
class Randomizer
{
 public:
    /** The constructor. */
    Randomizer(NodeCount noOfNodes)
        : _noOfNodes( noOfNodes ), _bijection( noOfNodes )
        {}

    /**
       Randomizes an Edge. The source and the target vertices are randomized
       using the bijection. The original source and target vertices are
       saved, so a RelabeledEdge, which contains both the randomized and the
       original vertices, is returned.
    */
    RelabeledEdge randomize(const Edge &edge) const {
        return RelabeledEdge(randomize(edge.source()),randomize(edge.target()),
                             edge.weight(),edge.source(),edge.target());
    }

 private:
    /** The number of nodes that specifies the domain and co-domain of the bijection. */
    NodeCount _noOfNodes;

    /** The underlying bijection. */
    Bijection _bijection;

    /** Randomizes a node ID using the underlying bijection. */
    NodeID randomize(NodeID nodeID) const {
        NodeID x = nodeID;

        // The application of the underlying bijection is repeated
```

```
        // until the result is in the correct co-domain.
        do {
            x = _bijection( x );
        } while (x >= _noOfNodes);

        return x;
    }

};

#endif // RANDOMIZER_H
```

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